Advanced Calculus: MATH 410

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1. Uniform Continuity

Uniform continuity is a very useful concept. Here we introduce it in the context of real-valued functions with domains in $\mathbb{R}$.

Definition 1.1. *Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be* uniformly continuous *over D provided that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in D$ one has*

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \epsilon \,.$$

This is a stronger concept than that of continuity over D. Indeed, a function $f : D \to \mathbb{R}$ is continuous over D provided for every $y \in D$ and every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in D$ one has

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \epsilon \,.$$

Here δ depends on y and ϵ ($\delta = \delta_{y,\epsilon}$), while in Definition 1.1 of uniform continuity δ depends only on ϵ ($\delta = \delta_\epsilon$). In other words, when f is uniformly continuous over D a δ_ϵ can be found that works uniformly for every $y \in D$ — hence, the terminology. We have obviously proved the following.

Proposition 1.1. *Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ be uniformly continuous over D. Then f is continuous over D.*

Remark: There is an important difference between continuity and uniform continuity. Continuity is defined to be a property of a function at a point. A function is then said to be continuous over a set if it is continuous at each point in the set. Uniform continuity is defined to be a property of a function over a set. It makes no sense to talk about a function being uniformly continuous at a single point.

1.1. **Some Uniformly Continuous Functions.**

We now show that there are many uniformly continuous functions. Recall that a function $f : D \to \mathbb{R}$ is Lipschitz continuous over D provided there exists an $L \geq 0$ such that for every $x, y \in D$ one has

$$|f(x) - f(y)| \leq L|x - y| \,.$$

The following should be pretty clear.

Proposition 1.2. *Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ be Lipschitz continuous over D. Then f is uniformly continuous over D.*

Proof: Let $\epsilon > 0$. Pick $\delta > 0$ so that $L\delta < \epsilon$. Then for every $x, y \in D$

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| \leq L|x - y| \leq L\delta < \epsilon\,.$$

□

There many uniformly continuous functions because there are many Lipschitz continuous functions. Recall we have shown that if D is either either (a, b), $[a, b)$, $(a, b]$ or $[a, b]$ for some $a < b$ while $f : D \to \mathbb{R}$ is continuous over D and differentiable over (a, b) with f' bounded then f is Lipschitz continuous over D with

$$L = \sup\{|f'(x)| \,:\, x \in (a, b)\}\,.$$

Hence, every such function is uniformly continuous.

While there are many uniformly continuous functions, there are also many functions that are not uniformly continuous.

Examples: The functions $f : \mathbb{R}_+ \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x}\,, \qquad f(x) = x^2\,, \qquad f(x) = \sin\left(\frac{1}{x}\right),$$

are not uniformly continuous. We will give one approach to showing this in the next section.

Notice here that the derivatives in the above examples are all unbounded over $\mathbb{R}_+$:

$$f'(x) = -\frac{1}{x^2}\,, \qquad f'(x) = 2x\,, \qquad f'(x) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right).$$

By Propostion 1.2 this must be the case for all differentiable functions defined over open intervals that are not uniformly continuous. However, as the following exercise shows, having an unbounded derivative does not imply that a differentiable function is not uniformly continuous.

Exercise: Show that the function $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = x^{\frac{1}{2}}$ is uniformly continous over $\mathbb{R}_+$. Hint: First establish the inequality

$$\left|y^{\frac{1}{2}} - x^{\frac{1}{2}}\right| \leq |y - x|^{\frac{1}{2}} \qquad \text{for every } x, y \in \mathbb{R}_+\,.$$

Exercise. Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1]$ if there exists a $C \in \mathbb{R}_+$ such that for every $x, y \in D$ one has

$$|f(x) - f(y)| \leq C\,|x - y|^{\alpha}\,.$$

Show that if $f : D \to \mathbb{R}$ is Hölder continuous of order α for some $\alpha \in (0, 1]$ then it is uniformly continuous over D.

1.2. **Sequence Characterization of Uniform Continuity.** The following theorem gives a characterization of uniform continuity in terms of sequences that is handy for showing that certain functions are not uniformly continuous.

Theorem 1.1. *Let $D \subset \mathbb{R}$. Then $f : D \to \mathbb{R}$ is uniformly continuous over D if and only if for every $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset D$ one has*

$$\lim_{n\to\infty}(x_n - y_n) = 0 \qquad \Longrightarrow \qquad \lim_{n\to\infty}\big(f(x_n) - f(y_n)\big) = 0\,.$$

Remark: This characterization is taken as the definition of uniform continuity in the text.

Remark: You can use this characterization to show that a given function $f : D \to \mathbb{R}$ is not uniformly continuous by starting with a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that $z_n \to 0$ as $n \to \infty$. Next, you seek a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ such that $\{x_n + z_n\}_{n\in\mathbb{N}} \subset D$ and

$$\lim_{n\to\infty}\big(f(x_n) - f(x_n + z_n)\big) \neq 0\,.$$

Upon setting $y_n = x_n + z_n$, you will have then found sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}} \subset D$ such that

$$\lim_{n\to\infty}(x_n - y_n) = 0 \qquad \text{and} \qquad \lim_{n\to\infty}\big(f(x_n) - f(y_n)\big) \neq 0\,.$$

Theorem 1.1 then implies the function f is not uniformly continuous over D.

Example: The function $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = 1/x$ is not uniformly continuous. Let $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $z_n \to 0$ as $n \to \infty$. Then for every $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ one has $\{x_n + z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and

$$f(x_n) - f(x_n + z_n) = \frac{1}{x_n} - \frac{1}{x_n + z_n} = \frac{z_n}{x_n(x_n + z_n)}\,.$$

If we choose $x_n = z_n$ for every $n \in \mathbb{N}$ then

$$f(x_n) - f(x_n + z_n) = \frac{1}{2z_n} \not\to 0 \qquad \text{as } n \to \infty\,.$$

Hence, f cannot be uniformly continuous over $\mathbb{R}_+$ by Theorem 1.1.

Example: The function $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = x^2$ is not uniformly continuous. Let $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $z_n \to 0$ as $n \to \infty$. Then for every $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ one has $\{x_n + z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and

$$f(x_n) - f(x_n + z_n) = x_n^2 - (x_n + z_n)^2 = -2x_n z_n - z_n^2\,.$$

If we choose $x_n = 1/z_n$ for every $n \in \mathbb{N}$ then

$$f(x_n) - f(x_n + z_n) = -2 - z_n^2 \not\to 0 \qquad \text{as } n \to \infty\,.$$

Hence, f cannot be uniformly continuous over $\mathbb{R}_+$ by Theorem 1.1.

Exercise: Show the function $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = \sin(1/x)$ is not uniformly continuous. Hint: Proceed as in the first example above, but choose a particular $\{z_n\}_{n\in\mathbb{N}}$ to simplify things.

Now let us turn to the proof of Theorem 1.1. The proof is similar to the proof of the characterization of continuity at a point in terms of convergent sequences.

Proof: ($\Longrightarrow$) Let $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset D$ such that

$$\lim_{n\to\infty} (x_n - y_n) = 0\,.$$

We need to show that

$$\lim_{n\to\infty} \big(f(x_n) - f(y_n)\big) = 0\,.$$

Let $\epsilon > 0$. Because f is uniformly continuous over D there exists $\delta > 0$ such that for every $x, y \in D$ one has

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \epsilon\,.$$

Because $(x_n - y_n) \to 0$ as $n \to \infty$, we know $|x_n - y_n| < \delta$ ultimately as $n \to \infty$. Because $|x_n - y_n| < \delta$ implies $|f(x_n) - f(y_n)| < \epsilon$, it follows that $|f(x_n) - f(y_n)| < \epsilon$ ultimately as $n \to \infty$. Because $\epsilon > 0$ was arbitrary, we have shown that $\big(f(x_n) - f(y_n)\big) \to 0$ as $n \to \infty$.

($\Longleftarrow$) Suppose f is not uniformly continuous over D. Then there exist $\epsilon_o > 0$ such that for every $\delta > 0$ there exists $x, y \in D$ such that

$$|x - y| < \delta \qquad \text{and} \qquad |f(x) - f(y)| \geq \epsilon_o\,.$$

Hence, for every $n \in \mathbb{N}$ there exists $x_n, y_n \in D$ such that

$$|x_n - y_n| < \frac{1}{2^n} \qquad \text{and} \qquad |f(x_n) - f(y_n)| \geq \epsilon_o\,.$$

Clearly, $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset D$ such that

$$\lim_{n\to\infty} (x_n - y_n) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \big(f(x_n) - f(y_n)\big) \neq 0\,.$$

But this contradicts the part of our hypothesis that requires that $\big(f(x_n) - f(y_n)\big) \to 0$ as $n \to \infty$. Therefore f must be uniformly continuous over D. $\square$

1.3. **Sequential Compactness and Uniform Continuity.** The following theorem shows that if D is closed and bounded then continuity implies uniform continuity. What lies behind this result is the fact that D is sequentially compact when it is closed and bounded.

Theorem 1.2. *Let $D \subset \mathbb{R}$ be closed and bounded. Let $f : D \to \mathbb{R}$ be continuous. Then f is uniformly continuous over D.*

Proof: We will establish the uniform continuity of f by using the characterization of Theorem 1.1. Let $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset D$ such that

$$\lim_{n\to\infty} (x_n - y_n) = 0\,.$$

We need to show that

$$\lim_{n\to\infty} \big(f(x_n) - f(y_n)\big) = 0\,.$$

Suppose not. Then there exists $\epsilon_o > 0$ such that

$$|f(x_n) - f(y_n)| \geq \epsilon_o \qquad \text{frequently}\,.$$

Hence, there exists subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}, \{y_{n_k}\}_{k\in\mathbb{N}} \subset D$ such that

$$\lim_{k\to\infty} (x_{n_k} - y_{n_k}) = 0\,,$$

and

$$\big|f(x_{n_k}) - f(y_{n_k})\big| \geq \epsilon_o \qquad \text{for every } k \in \mathbb{N}\,. \tag{1}$$

Because D is sequentially compact, the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ has a further subsequence $\{x_{n_{k_l}}\}_{l\in\mathbb{N}}$ that converges to some $x_* \in D$. Because

$$\lim_{l\to\infty} \big(y_{n_{k_l}} - x_{n_{k_l}}\big) = 0\,,$$

we see that $\{y_{n_{k_l}}\}_{l\in\mathbb{N}}$ also converges with

$$\lim_{l\to\infty} y_{n_{k_l}} = \lim_{l\to\infty} x_{n_{k_l}} + \lim_{l\to\infty} \big(y_{n_{k_l}} - x_{n_{k_l}}\big) = x_* + 0 = x_*\,.$$

Because f is continuous at $x_* \in D$, we know that

$$\lim_{l\to\infty} \big(f(x_{n_{k_l}}) - f(y_{n_{k_l}})\big) = f(x_*) - f(x_*) = 0\,.$$

But this contradicts our supposition, which by (1) implies that

$$\big|f(x_{n_{k_l}}) - f(y_{n_{k_l}})\big| \geq \epsilon_o \qquad \text{for every } l \in \mathbb{N}\,.$$

Therefore

$$\lim_{n\to\infty} \big(f(x_n) - f(y_n)\big) = 0\,,$$

whereby f is uniformly continuous by Theorem 1.1. □

The conclusion of the above theorem can still hold for some cases where D is closed but unbounded. For example, if $D = \mathbb{Z}$ then every function is uniformly continuous. This is easily seen from the definition by taking $\delta < 1$. However, the next proposition shows that the hypothesis D is closed cannot be dropped.

Proposition 1.3. *Let $D \subset \mathbb{R}$. If D is not closed then there exists a function $f : D \to \mathbb{R}$ that is continuous over D, but that is not uniformly continuous over D.*

Proof: Because D is not closed there exists a limit point x_* of D that is not in D. Consider the function $f : D \to \mathbb{R}$ defined for every $x \in D$ by $f(x) = 1/(x - x_*)$. It should be clear to you that this function is continuous over D. We will show that it is not uniformly continuous over D by showing that for every $\delta > 0$ there exists $x, y \in D$ such that

$$|x - y| < \delta\,, \qquad \text{and} \qquad |f(x) - f(y)| \geq 1\,.$$

Because x_* is a limit point of D there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ such that $x_n \to x_*$ as $n \to \infty$. Let $\delta > 0$ be arbitrary. Let $m \in \mathbb{N}$ such that

$$n \geq m \qquad \Longrightarrow \qquad |x_n - x_*| < \frac{\min\{\delta, 1\}}{2}\,.$$

Then for every $k \in \mathbb{N}$ one has

$$|x_{m+k} - x_m| \leq |x_{m+k} - x_*| + |x_m - x_*| < \frac{\delta}{2} + \frac{\delta}{2} = \delta\,,$$

while

$$|x_{m+k} - x_*| < \frac{1}{2}\,.$$

This last inequality implies that for every $k \in \mathbb{N}$ one has

$$|f(x_{m+k}) - f(x_m)| = \frac{|x_m - x_{m+k}|}{|x_{m+k} - x_*|\,|x_m - x_*|} \geq 2\,\frac{|x_m - x_{m+k}|}{|x_m - x_*|}\,.$$

Because

$$\lim_{k\to\infty} \frac{|x_m - x_{m+k}|}{|x_m - x_*|} = 1\,,$$

we may pick a $k \in \mathbb{N}$ large enough so that

$$\frac{|x_m - x_{m+k}|}{|x_m - x_*|} > \frac{1}{2}\,.$$

Then for this k we have

$$|x_{m+k} - x_m| < \delta\,, \quad \text{and} \quad |f(x_{m+k}) - f(x_m)| \geq 2\frac{|x_m - x_{m+k}|}{|x_m - x_*|} > 2\frac{1}{2} = 1\,.$$

Because $\delta > 0$ was arbitrary, we can conclude that f is not uniformly continuous over D. $\square$