Final Exam Solutions: MATH 410
Saturday, 16 December 2006

1. [30] State whether each of the following statements is true or false. Give a proof when true and a counterexample when false.

   (a) A sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ is convergent if the sequence $\{a_k^2\}_{k\in\mathbb{N}}$ is convergent.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ then $f' > 0$ over $\mathbb{R}$.

   (c) A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable over $[a, b]$ if the function f^2 is is Riemann integrable over $[a.b]$.

   Solution (a): This is *false*. A simple counterexample is given by $a_k = (-1)^k$ for every $k \in \mathbb{N}$. Then the sequence $\{a_k^2\}_{k\in\mathbb{N}}$ converges to 1 (because $a_k^2 = (-1)^{2k} = 1$), while the sequence $\{a_k\}_{k\in\mathbb{N}}$ does not converge. □

   Solution (b): This is also *false*. A simple counterexample is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. This function is clearly increasing and differentiable over $\mathbb{R}$ with $f'(x) = 3x^2$. Hence, $f'(0) = 0$, which is not positive.

   Solution (c): This is also *false*. A simple counterexample is $f : [a, b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\,, \\ -1 & \text{otherwise}\,. \end{cases}$$

   The function f^2 is is Riemann integrable over $[a.b]$ because $f^2(x) = (f(x))^2 = 1$ for every $x \in [a, b]$, while f is not Riemann integrable over $[a.b]$ because $\overline{L}(f) = -1 < 1 = \underline{U}(f)$. Indeed, for every partition P of $[a, b]$ one has $L(f, P) = -1$ and $U(f, P) = 1$.

2. [20] Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$.

   (a) Prove that

$$\limsup_{k\to\infty}(a_k + b_k) \leq \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k\,.$$

   (b) Give an example for which equality does not hold above.

Solution (a): Let $c_k = a_k + b_k$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ we define

$$\begin{aligned} \overline{a}_k &= \sup\{a_l \,:\, l \geq k\}\,, \\ \overline{b}_k &= \sup\{b_l \,:\, l \geq k\}\,, \\ \overline{c}_k &= \sup\{c_l \,:\, l \geq k\}\,. \end{aligned}$$

Because $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are bounded above, for every $k \in \mathbb{N}$ we have

$$\overline{a}_k < \infty\,, \qquad \overline{b}_k < \infty\,, \qquad \overline{c}_k < \infty\,.$$

Therefore $\{\overline{a}_k\}_{k\in\mathbb{N}}$, $\{\overline{b}_k\}_{k\in\mathbb{N}}$, and $\{\overline{b}_k\}_{k\in\mathbb{N}}$ are nonincreasing sequences in $\mathbb{R}$. Moreover, because $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, and $\{c_k\}_{k\in\mathbb{N}}$ are bounded below, the sequences $\{\overline{a}_k\}_{k\in\mathbb{N}}$, $\{\overline{b}_k\}_{k\in\mathbb{N}}$, and $\{\overline{b}_k\}_{k\in\mathbb{N}}$ are also bounbed below. They therefore converge by Montonic Sequence Convergence Theorem. By the definition of lim sup we have

$$\begin{aligned} \limsup_{k\to\infty} a_k &= \lim_{k\to\infty} \overline{a}_k\,, \\ \limsup_{k\to\infty} b_k &= \lim_{k\to\infty} \overline{b}_k\,, \\ \limsup_{k\to\infty} c_k &= \lim_{k\to\infty} \overline{c}_k\,. \end{aligned}$$

The key step is to prove that $\overline{c}_k \leq \overline{a}_k + \overline{b}_k$ for every $k \in \mathbb{N}$. Because for every $k \in \mathbb{N}$ we have

$$a_l \leq \overline{a}_k\,, \quad \text{and} \quad b_l \leq \overline{b}_k\,, \quad \text{for every } l \geq k\,,$$

it follows that for every $k \in \mathbb{N}$ we have

$$c_l = a_l + b_l \leq \overline{a}_k + \overline{b}_k \quad \text{for every } l \geq k\,.$$

Hence,

$$\overline{c}_k = \sup\{c_l \,:\, l \geq k\} \leq \overline{a}_k + \overline{b}_k\,.$$

Then by the properties of limits

$$\begin{aligned} \limsup_{k\to\infty}(a_k + b_k) &= \limsup_{k\to\infty} c_k \\ &= \lim_{k\to\infty} \overline{c}_k \\ &\leq \lim_{k\to\infty} (\overline{a}_k + \overline{b}_k) \\ &= \lim_{k\to\infty} \overline{a}_k + \lim_{k\to\infty} \overline{b}_k \\ &= \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k\,. \quad \square \end{aligned}$$

Solution (b): Let $a_k = (-1)^k$ and $b_k = (-1)^{k+1}$ for every $k \in \mathbb{N}$. Clearly

$$\limsup_{k\to\infty} a_k = \lim_{k\to\infty} a_{2k} = 1\,,$$
$$\limsup_{k\to\infty} b_k = \lim_{k\to\infty} b_{2k+1} = 1\,,$$

while (because $a_k + b_k = 0$ for every $k \in \mathbb{N}$)

$$\limsup_{k\to\infty}(a_k + b_k) = \lim_{k\to\infty}(a_k + b_k) = 0\,.$$

Therefore

$$\limsup_{k\to\infty}(a_k + b_k) = 0 < 2 = \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k\,. \quad \square$$

3. [20] Determine all $a \in \mathbb{R}$ for which the following formal infinite series converge. Give your reasoning.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log(n)}\, a^n$

Solution: The series converges for $a \in [-1, 1)$ and diverges otherwise.

The cases $|a| < 1$ and $|a| > 1$ are best handled by the Ratio Test. Let $b_n = a^n/\log(n)$. Because

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{\log(n+1)}{\log(n)}|a| = |a|\,,$$

the Ratio Test then implies that this series converges absolutely for $|a| < 1$ and diverges for $|a| > 1$.

The case $a = -1$ is best handled by the Alternating Series Test. Indeed, because the sequence

$$\left\{\frac{1}{\log(n)}\right\}_{n=2}^{\infty} \quad \text{is decreasing and positive}\,.$$

and because

$$\lim_{n\to\infty} \frac{1}{\log(n)} = 0\,,$$

the Alternating Series Test shows that

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log(n)} \quad \text{converges}\,.$$

The case $a = 1$ is best handled by Limit Comparison Test, say with the harmonic series. Indeed, because

$$\lim_{n\to\infty} \frac{\log(n)}{n} = 0\,,$$

and because the harmonic series

$$\sum_{n=2}^{\infty} \frac{1}{n} \quad \text{diverges}\,,$$

the Limit Comparison Test shows that

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)} \quad \text{diverges}\,.$$

Alternatively, one could treat this case with the Direct Comparison Test, the Integral Test, or the Cauchy 2^k Test. □

(b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k^5+1}\right)^a$

Solution: The series converges for $a \in (\frac{1}{4}, \infty)$ and diverges otherwise. Because

$$\frac{k}{k^5+1} \sim \frac{1}{k^4} \quad \text{as } k \to \infty\,,$$

one sees that the original series should be compared with the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^{4a}}\,.$$

This is best handled by Limit Comparison Test. Indeed, because for every $a \in \mathbb{R}$ one has

$$\lim_{k\to\infty} \frac{\left(\dfrac{k}{k^5+1}\right)^a}{\dfrac{1}{k^{4a}}} = \lim_{k\to\infty} \left(\frac{k^5}{k^5+1}\right)^a = 1\,,$$

the Limit Comparison Test then implies that

$$\sum_{k=1}^{\infty} \left(\frac{k}{k^5+1}\right)^a \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k^{4a}} \quad \text{converges}\,.$$

Because $p = 4a$ for the p-series, that series converges for $a \in (\frac{1}{4}, \infty)$ and diverges otherwise. The same is therefore true for the original series. □

4. [20] Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous over (a, b). Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in (a, b). Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence.

   Solution: Let $\epsilon > 0$. Because $f : (a, b) \to \mathbb{R}$ is uniformly continuous over (a, b), there exists a $\delta > 0$ such that for all points $x, y \in (a, b)$ one has

   $$|x - y| < \delta \quad \Longrightarrow \quad \big|f(x) - f(y)\big| < \epsilon\,.$$

   Because $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for every $k, l \in \mathbb{N}$ one has

   $$k, l > N \quad \Longrightarrow \quad |x_k - x_l| < \delta\,.$$

   Hence, for every $k, l \in \mathbb{N}$ one has

   $$\begin{aligned} k, l > N \quad &\Longrightarrow \quad |x_k - x_l| < \delta \\ &\Longrightarrow \quad \big|f(x_k) - f(x_l)\big| < \epsilon\,. \end{aligned}$$

   Therefore $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. □

5. [10] Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in (a, b). Give negations of each of the following assertions.

   (a) For every $\epsilon > 0$ there exists an $n_\epsilon \in \mathbb{N}$ such that

   $$m, n > n_\epsilon \Longrightarrow |x_m - x_n| < \epsilon\,.$$

   Solution: There exists $\epsilon > 0$ such that for every $l \in \mathbb{N}$ there exists $m, n \in \mathbb{N}$ such that

   $$m, n > l \qquad \text{and} \qquad |x_m - x_n| \geq \epsilon\,.$$

   (b) There exists a $c \in \mathbb{R}$ such that no subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to c.

   Solution: For every $c \in \mathbb{R}$ there exists a subsequence of $\{x_n\}_{n=1}^{\infty}$ that converges to c.

6. [20] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) < 0$. Show that there exists a $\delta > 0$ such that

   $$\begin{aligned} x \in (c - \delta, c) \subset (a, b) &\Longrightarrow f(x) > f(c)\,, \\ x \in (c, c + \delta) \subset (a, b) &\Longrightarrow f(c) > f(x)\,, \end{aligned}$$

   Remark: It is very incorrect to assert that f is decreasing in an interval containing c.

Solution: Because f is differentiable at c, we have

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)\,.$$

Because $f'(c) < 0$ there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset (a,b)$ and

$$\begin{aligned} 0 < |x-c| < \delta \qquad &\Longrightarrow \qquad \left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < -f'(c) \\ &\Longrightarrow \qquad \frac{f(x)-f(c)}{x-c} < 0\,. \end{aligned}$$

Hence,

$$\begin{aligned} x \in (c-\delta, c) &\Longrightarrow f(x)-f(c) = \frac{f(x)-f(c)}{x-c}(x-c) > 0 \\ &\Longrightarrow f(x) > f(c)\,, \\ x \in (c, c+\delta) &\Longrightarrow f(x)-f(c) = \frac{f(x)-f(c)}{x-c}(x-c) < 0 \\ &\Longrightarrow f(x) < f(c)\,. \end{aligned}$$

□

7. [20] Let $f(x) = \sinh(x) \equiv \frac{1}{2}(e^x - e^{-x})$ for every $x \in \mathbb{R}$. Then for every $k \in \mathbb{N}$ and every $x \in \mathbb{R}$ one has

$$f^{(2k)}(x) = \sinh(x)\,, \qquad f^{(2k+1)}(x) = \cosh(x)\,.$$

Show that

$$\sinh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\, x^{2k+1} \quad \text{for every } x \in \mathbb{R}\,.$$

Solution: Because $f(x) = \sinh(x) \equiv \frac{1}{2}(e^x - e^{-x})$, we have $\cosh(x) = f'(x) = \frac{1}{2}(e^x + e^{-x})$. It follows that

$$f^{(2k)}(0) = \sinh(0) = 0\,, \qquad f^{(2k+1)}(0) = \cosh(0) = 1\,.$$

The series is therefore just the formal Taylor series for f centered at 0. Moreover, we see that the n^{th} partial sum can be expressed as a Taylor polynomial approximation in two ways:

$$\sum_{k=0}^{n} \frac{1}{(2k+1)!}\, x^{2k+1} = T_0^{(2n+1)} \sinh(x) = T_0^{(2n+2)} \sinh(x)\,.$$

If we use the last expression, the Lagrange Remainder Theorem then states that for every nonzero $x \in \mathbb{R}$

$$\sinh(x) = T_0^{(2n+2)} \sinh(x) + \frac{1}{(2n+3)!} \cosh(p) x^{2n+3} ,$$

for some p between 0 and x. Because cosh is an even function that is increasing over $[0, \infty)$, for every p between 0 and x one has $\cosh(p) < \cosh(x)$. Hence, for every $x \in \mathbb{R}$

$$\left| \sinh(x) - \sum_{k=0}^{n} \frac{1}{(2k+1)!} x^{2k+1} \right| \leq \frac{1}{(2n+3)!} \cosh(x) |x|^{2n+3} .$$

Because for every $x \in \mathbb{R}$

$$\lim_{n \to \infty} \frac{1}{(2n+3)!} \cosh(x) |x|^{2n+3} = 0 ,$$

the sequence of partial sums therefore converges to $\sinh(x)$. $\square$

Remark: An alternative approach is to first show that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for every } x \in \mathbb{R} .$$

and then use the fact $f(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$ to derive the series for sinh. The first step uses the Lagrange Remaninder Theorem and is given in the notes while second goes like

$$\begin{aligned} \sinh(x) &= \tfrac{1}{2}(e^x - e^{-x}) \\ &= \tfrac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n - \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1-(-1)^n}{2} x^n \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} . \end{aligned}$$

$\square$

8. [20] Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Show that for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$0 \leq U(f, P) - L(f, P) < \epsilon ,$$

where $L(f, P)$ and $U(f, P)$ are the lower and upper Darboux sums associated with f and P.

Solution: Let $\epsilon > 0$. Because

$$\overline{L}(f) = \sup\{L(f,P) \,:\, P \text{ is a partition of } [a,b]\}\,,$$
$$\underline{U}(f) = \inf\{U(f,P) \,:\, P \text{ is a partition of } [a,b]\}\,,$$

there exists partitions P^L and P^U of $[a,b]$ such that

$$\overline{L}(f) - \frac{\epsilon}{2} < L(f,P^L) \leq \overline{L}(f)\,,$$
$$\underline{U}(f) \leq U(f,P^U) < \underline{U}(f) + \frac{\epsilon}{2}\,.$$

Let $P^\epsilon = P^L \vee P^U$. Then by the Refinement Lemma

$$\overline{L}(f) - \frac{\epsilon}{2} < L(f,P^L) \leq L(f,P^\epsilon) \leq \overline{L}(f)\,,$$
$$\underline{U}(f) \leq U(f,P^\epsilon) \leq U(f,P^U) < \underline{U}(f) + \frac{\epsilon}{2}\,.$$

Because f is Riemann integrable, $\overline{L}(f) = \underline{U}(f)$. Hence,

$$0 \leq U(f,P^\epsilon) - L(f,P^\epsilon) < \left(\underline{U}(f) + \frac{\epsilon}{2}\right) - \left(\overline{L}(f) - \frac{\epsilon}{2}\right) = \epsilon\,.$$

□

9. [20] Let $f : [a,b] \to \mathbb{R}$ be continuous. Prove that there exists $p \in (a,b)$ such that

$$f(p) = \frac{1}{e^b - e^a} \int_a^b f(x) e^x \, \mathrm{d}x\,.$$

Solution: Let $g : [a,b] \to \mathbb{R}$ be given by $g(x) = e^x$ for every $x \in [a,b]$. Clearly g is Riemann integrable over $[a,b]$. Because $f : [a,b] \to \mathbb{R}$ is continuous while $g : [a,b] \to \mathbb{R}$ is positive and Riemann integrable, the Integral Mean-Value Theorem implies there exists $p \in (a,b)$ such that

$$\int_a^b f(x) g(x) \, \mathrm{d}x = f(p) \int_a^b g(x) \, \mathrm{d}x\,.$$

But

$$\int_a^b g(x) \, \mathrm{d}x = \int_a^b e^x \, \mathrm{d}x = e^b - e^a > 0\,,$$

so that

$$f(p) = \frac{1}{e^b - e^a} \int_a^b f(x) e^x \, \mathrm{d}x\,.$$

□

10. [20] Prove that every countable set has measure zero.

Solution: Let $A \subset \mathbb{R}$ be countable. Let $\epsilon > 0$. Because A is countable there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $A \subset \{x_k\}_{k\in\mathbb{N}}$. Let $r < \frac{1}{2}$. Then

$$A \subset \{x_k\}_{k\in\mathbb{N}} \subset \bigcup_{k\in\mathbb{N}} (x_k - r^{k+2}\epsilon, x_k + r^{k+2}\epsilon) \,,$$

while (because $r < \frac{1}{2}$ implies $2r^2/(1-r) < 1$)

$$\sum_{k=0}^{\infty} 2r^{k+2}\epsilon = \frac{2r^2\epsilon}{1-r} < \epsilon \,.$$

But $\epsilon > 0$ was arbitrary, so A has measure zero. □