Second In-Class Exam Solutions: MATH 410
Monday, 20 November 2006

1. [20] State whether each of the following statements is true or false. Give a proof when true and a counterexample when false.

   (a) If $f : (a, b) \to \mathbb{R}$ is continuous then f has a minimum or a maximum over (a, b).

   Solution: This is *false*. A simple counterexample is $f : (a, b) \to \mathbb{R}$ given by $f(x) = x$. This function is clearly continuous over (a, b). The range of f is (a, b), which does not have either a minimum or a maximum.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ then $f' > 0$ over $\mathbb{R}$.

   Solution: This is also *false*. A simple counterexample is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. This function is clearly increasing and differentiable over $\mathbb{R}$ with $f'(x) = 3x^2$. Hence, $f'(0) = 0$, which is not positive.

2. [20] Determine all $a \in \mathbb{R}$ for which the following formal infinite series converge. Give your reasoning.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log(n)}\, a^n$

   Solution: The series converges for $a \in [-1, 1)$ and diverges otherwise.

   The cases $|a| < 1$ and $|a| > 1$ are best handled by the Ratio Test. Let $b_n = a^n / \log(n)$. Because

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \frac{\log(n+1)}{\log(n)} |a| = |a| \,,$$

   the Ratio Test then implies that this series converges absolutely for $|a| < 1$ and diverges for $|a| > 1$.

   The case $a = -1$ is best handled by the Alternating Series Test. Indeed, because the sequence

$$\left\{ \frac{1}{\log(n)} \right\}_{n=2}^{\infty} \quad \text{is decreasing and positive}\,.$$

   and because

$$\lim_{n\to\infty} \frac{1}{\log(n)} = 0 \,,$$

the Alternating Series Test shows that

$$\sum_{n=2}^{\infty}(-1)^n \frac{1}{\log(n)} \quad \text{converges}\,.$$

The case $a = 1$ is best handled by Limit Comparison Test, say with the harmonic series. Indeed, because

$$\lim_{n\to\infty} \frac{\log(n)}{n} = 0\,,$$

and because the harmonic series

$$\sum_{n=2}^{\infty} \frac{1}{n} \quad \text{diverges}\,,$$

the Limit Comparison Test shows that

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)} \quad \text{diverges}\,.$$

Alternatively, one could treat this case with the Integral Test or the Cauchy 2^k Test. □

(b) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{k^5+1}\right)^a$

Solution: The series converges for $a \in [\frac{1}{4}, \infty)$ and diverges otherwise. Because

$$\frac{k}{k^5+1} \sim \frac{1}{k^4} \quad \text{as } k \to \infty\,,$$

one sees that the original series should be compared with the p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^{4a}}\,.$$

This is best handled by Limit Comparison Test. Indeed, because for every $a \in \mathbb{R}$ one has

$$\lim_{k\to\infty} \frac{\left(\dfrac{k}{k^5+1}\right)^a}{\dfrac{1}{k^{4a}}} = \lim_{k\to\infty}\left(\frac{k^5}{k^5+1}\right)^a = 1\,,$$

the Limit Comparison Test then implies that

$$\sum_{k=1}^{\infty}\left(\frac{k}{k^5+1}\right)^a \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty}\frac{1}{k^{4a}} \quad \text{converges}\,.$$

Because the $p = 4a$ for the p-series, it converges for $a \in (\frac{1}{4}, \infty)$ and diverges otherwise. The same is therefore true for the original series. □

3. [10] Prove that for every $x > -1$ one has

$$(1+x)^{\frac{1}{4}} \leq 1 + \tfrac{1}{4}x\,.$$

Solution: One approach to this problem uses the Lagrange Remainder Theorem. Define $f(x) = (1+x)^{\frac{1}{4}}$ for every $x > -1$. Then

$$f'(x) = \tfrac{1}{4}(1+x)^{-\frac{3}{4}}\,, \qquad f''(x) = -\tfrac{3}{16}(1+x)^{-\frac{7}{4}}\,.$$

By the Lagrange Remainder Theorem there exists a p between 0 and x such that

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(p)x^2\,.$$

Hence,

$$(1+x)^{\frac{1}{4}} - 1 - \tfrac{1}{4}x = -\tfrac{3}{16}(1+p)^{-\frac{7}{4}}x^2 \leq 0\,.$$

The result follows. □

A second approach to this problem uses the Monotonicity Theorem. Define $g(x) = (1+x)^{\frac{1}{4}} - 1 - \frac{1}{4}x$ for every $x > -1$. Then

$$g'(x) = \tfrac{1}{4}\big[(1+x)^{-\frac{3}{4}} - 1\big]\,.$$

Clearly, $g'(x) > 0$ for $x \in (-1, 0)$ while $g'(x) < 0$ for $x \in (0, \infty)$. By the Monotonicity Theorem, g is increasing over $x \in (-1, 0]$ and g is decreasing over $[0, \infty)$. Therefore 0 is a global maximizer of g over $(-1, \infty)$, and $g(0) = 0$ is the maximum of g over $(-1, \infty)$. Hence, for every $x > -1$ we have

$$(1+x)^{\frac{1}{4}} - 1 - \tfrac{1}{4}x = g(x) \leq g(0) = 0\,.$$

The result follows. □

A third approach to this problem is based on the observation that becauce $t \mapsto t^{\frac{1}{4}}$ is an increasing function of t over $[0, \infty)$, the result follows if we can show that

$$1 + x \leq (1 + \tfrac{1}{4}x)^4 \quad \text{for every } x > -1\,.$$

But a direct calculation (assuming $1+x>0$) shows that

$$\begin{aligned}(1+\tfrac{1}{4}x)^4 &= 1+x+\tfrac{6}{4^2}x^2+\tfrac{1}{4^2}x^3+\tfrac{1}{4^4}x^4\\ &= 1+x+\tfrac{1}{4^2}x^2\big(6+x+\tfrac{1}{4^2}x^2\big)\\ &\geq 1+x+\tfrac{1}{4^2}x^2\big(5+\tfrac{1}{4^2}x^2\big)\\ &\geq 1+x\,.\end{aligned}$$

The result follows. □

4. [20] Evaluate the following limits. Give your reasoning.

$$\text{(a)}\quad \lim_{x\to 1}\frac{x^4-1}{x-1}\,, \qquad\qquad \text{(b)}\quad \lim_{x\to 2}\frac{x^4-1}{x-1}\,.$$

Solution: For every $x\neq 1$ one has

$$\frac{x^4-1}{x-1}=1+x+x^2+x^3\,.$$

Because the right-hand side above is continuous over $\mathbb{R}$, one has

$$\begin{aligned}&\text{(a)}\quad \lim_{x\to 1}\frac{x^4-1}{x-1}=\lim_{x\to 1}\big(1+x+x^2+x^3\big)=4\,,\\ &\text{(b)}\quad \lim_{x\to 2}\frac{x^4-1}{x-1}=\lim_{x\to 2}\big(1+x+x^2+x^3\big)=15\,.\end{aligned}$$

5. [20] Consider the formal infinite series

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\,x^{2k+1}\,.$$

(a) Determine all $x\in\mathbb{R}$ for which this series converges.
(b) Show that if this series converges for some $x\in\mathbb{R}$ then it sums to $\sin(x)$.

Solution: The series converges to $\sin(x)$ for every $x\in\mathbb{R}$. Indeed, let $f(x)=\sin(x)$. Then for every $k\in\mathbb{N}$ one has

$$f^{(2k)}(x)=(-1)^k\sin(x)\,,\qquad f^{(2k+1)}(x)=(-1)^k\cos(x)\,.$$

Because

$$f^{(2k)}(0)=0\,,\qquad f^{(2k+1)}(0)=(-1)^k\,,$$

the series is just the formal Taylor series for f centered at 0. Moreover, we see that the n^{th} partial sum can be expressed as a Taylor polynomial approximation in two ways:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = T_0^{(2n+1)} \sin(x) = T_0^{(2n+2)} \sin(x)\,.$$

Using the last expression, the Lagrange Remainder Theorem states that for every $x \in \mathbb{R}$

$$\sin(x) = T_0^{(2n+2)} \sin(x) + \frac{(-1)^{n+1}}{(2n+3)!} \cos(p) x^{2n+3}\,,$$

for some p between 0 and x. Hence, for every $x \in \mathbb{R}$

$$\left| \sin(x) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \right| \leq \frac{1}{(2n+3)!} |x|^{2n+3}\,.$$

Because for every $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{(2n+3)!} |x|^{2n+3} = 0\,,$$

the sequence of partial sums therefore converges to $\sin(x)$. □

6. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let c be a limit point of D. Write negations of the following assertions.

(a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ one has

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} f(x_k) = \infty\,."$$

Solution: There exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset D - \{c\}$ such that

$$\lim_{k\to\infty} |x_k - c| = 0 \quad \text{and} \quad \liminf_{k\to\infty} f(x_k) < \infty\,.$$

(b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ one has

$$0 < |x - c| < \delta \quad \Longrightarrow \quad f(x) > M\,."$$

Solution: There exists $M \in \mathbb{R}$ such that for every $\delta > 0$ there exists $x \in D$ such that

$$0 < |x - c| < \delta \quad \text{and} \quad f(x) \leq M\,.$$