Sixth Homework: MATH 410
Due in class, Monday, 16 October 2006

1. Find a positive sequence $\{a_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} a_k = 0\,, \quad \text{and} \quad \sum_{k=0}^{\infty}(-1)^k a_k \quad \text{diverges}\,.$$

Remark: Such examples show that one cannot simply drop the "nonincreasing" hypothesis from the Alternating Series Test.

2. Let $\{a_k\}_{k\in\mathbb{N}}$ be a nonzero real sequence such that

$$\frac{|a_{k+1}|}{|a_k|} \geq 1 \quad \text{ultimately as } k \to \infty\,.$$

Show that

$$\sum_{k=0}^{\infty} a_k \quad \text{diverges}\,.$$

Remark: This is the "divergence" conclusion of the Ratio Test.

3. Let $\{a_k\}_{k\in\mathbb{N}}$ be a nonzero real sequence.
   (a) Prove that

$$\limsup_{k\to\infty} |a_k|^{\frac{1}{k}} \leq \limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}\,.$$

   (b) Find an example for which the above inequality is strict with the left-hand side less than 1 and the right-hand side greater than 1.

Remark: This problem shows that the convergence assertion made by the Root Test is generally sharper than the one made by the Ratio Test.

4. Determine all the values of $x \in \mathbb{R}$ for which the formal infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin(nx) \quad \text{converges}\,.$$

Give your reasoning.