1. For $f \in C^m$, $Q(x^k) = I(x^k)$, $\forall\ k < m$

a)

$$E(f) = \int_{x_L}^{x_R} k_m(x) f^{(m)}(x)\, dx$$

where $k_m(x) = \frac{1}{(m-1)!} \int_{x_L}^{x_R} (z - x)^{m-1} w(z)\, dz$

$$- \frac{1}{(m-1)!} \sum_{j=1}^{n} (x_j - x)^{m-1} H(x - x_j)\, w_j$$

$H(x)$ is heaviside function.

We know for Linear function.

$$Q(x) = I(x)$$

For Mid-pt rule

$$x_j = \frac{x_{i-1} + x_i}{2}, \qquad w(x) = 1$$

$$k_2(x) = \int_{x_{i-1}}^{x} (z - x)\, dz - \Big(\frac{x_i + x_{i-1}}{2} - x\Big) \underbrace{}_{} H\Big(x - \frac{x_i + x_{i-1}}{2}\Big) \quad (x_i - x_{i-1})$$

$$= -\frac{(x_{i-1} - x)^2}{2} - (x_i^2 - x_{i-1}^2 - 2x x_i - 2x x_{i-1})/2 \cdot H\Big(x - \frac{x_{i-1} + x_i}{2}\Big)$$

① If $x \le \frac{x_{i-1} + x_i}{2}$, $H\Big(x - \frac{x_{i-1} + x_i}{2}\Big) = 0$

$$\therefore\ k_2(x) = -\frac{(x_{i-1} - x)^2}{2}$$

② If $x > \frac{x_{i-1} + x_i}{2}$, $H\Big(x - \frac{x_{i-1} + x_i}{2}\Big) = 1$

$$k_2(x) = -\frac{(x_i - x)^2}{2}$$

$k_2(x)$ is always nonpositive. $\min\limits_{x} |k_2(x)| = \frac{\Delta^2}{8}$

b) $$E_n(f) = \int_{x_L}^{x_R} k(x)\, f''(x)\, dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} k(x) f''(x) dx$$

$$\int_{x_{i-1}}^{x_i} k(x)\, dx = \int_{x_{i-1}}^{\frac{x_i + x_{i-1}}{2}} -\frac{(x - x_{i-1})^2}{2} dx + \int_{\frac{x_i + x_{i-1}}{2}}^{x_i} -\frac{(x_i - x)^2}{2} dx$$

$$= -\frac{\Delta^3}{24}$$

$$E_n(f) \le \|f''\|_\infty \left| \sum_i \left(-\frac{\Delta^3}{24}\right) \right|$$

$$= \frac{\Delta^2}{24} (x_R - x_L) \|f''\|_\infty$$

And we have equality when $f'' = C$.
so, this is a sharp bound.

c) since $\|K(x)\|_\infty = \frac{\Delta^2}{8}$

$$|E_n(f)| = \left| \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} K(x) f''(x) dx \right|$$

$$\le \frac{\Delta^2}{8} \int_{x_L}^{x_R} f''(x) dx$$

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Let $Q_\Delta(f) =$ Quadrature using mid-pt rule

$T_\Delta(f) =$ Quadrature using trapezoid rule

Since
$$T_\Delta(f) = I(f) + \alpha_2 \Delta^2 + O(\Delta^4)$$
$$T_{\frac{\Delta}{2}}(f) = I(f) + \frac{\alpha_2}{4}\Delta^2 + O(\Delta^4)$$

$$\frac{1}{2}\left(T_\Delta(f) + Q_\Delta(f)\right) = T_{\frac{\Delta}{2}}(f)$$

so,
$$Q_\Delta(f) = \alpha T_{\frac{\Delta}{2}}(f) - T_\Delta(f)$$
$$= I(f) - \frac{\alpha_2}{2}\Delta^2 + O(\Delta^4)$$

where
$$\alpha_2 = \frac{1}{12}\left[f'(x_R) - f'(x_L)\right]$$

$$\therefore \quad Q(f) = I(f) - \frac{\Delta^2}{24}\left[f'(x_R) - f'(x_L)\right] + O(\Delta^4)$$

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3. By Euler - Maclaurin asymptotic formula:

$$Q_{\Delta}(f) = I(f) + \alpha_2 \Delta^2 + \alpha_4 \Delta^4 + \alpha_6 \Delta^6 + O(\Delta^8)$$

$$Q_{2\Delta}(f) = I(f) + 4\alpha_2 \Delta^2 + 16\alpha_4 \Delta^4 + 64\alpha_6 \Delta^6 + O(\Delta^8)$$

$$Q_{3\Delta}(f) = I(f) + 9\alpha_2 \Delta^2 + 81\alpha_4 \Delta^4 + 729\alpha_6 \Delta^6 + O(\Delta^8)$$

$$Q_{6\Delta}(f) = I(f) + 36\alpha_2 \Delta^2 + 1296\alpha_4 \Delta^4 + 46656\Delta^6 + O(\Delta^8)$$

Let $Q(f) = w_1 Q_{\Delta}(f) + w_2 Q_{2\Delta}(f) + w_3 Q_{3\Delta}(f) + w_4 Q_{6\Delta}(f$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 9 & 36 \\ 1 & 16 & 81 & 1296 \\ 1 & 64 & 729 & 46656 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$w_1 = \frac{1296}{840}$, $w_2 = -\frac{567}{840}$, $w_3 = \frac{112}{840}$, $w_4 = -\frac{1}{84}$

$$Q(f) = \sum_{i=1}^{4} w_i Q_{i\Delta}(f).$$

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4. Note $\int_0^\infty x^n e^{-x} = n!$, $P(x)$ monic.

$P_0(x) = 1$,

$$\mu_0 = \frac{(xP_0|P_0)}{(P_0|P_0)} = \frac{\int_0^\infty x e^{-x} dx}{\int_0^1 e^{-x} dx} = 1$$

$P_1(x) = x - \mu_0 = x - 1$

$$\mu_1 = \frac{(xP_1|P_1)}{(P_1|P_1)} = \frac{\int_0^\infty x(x-1)^2 e^{-x} dx}{\int_0^\infty (x-1)^2 e^{-x} dx} = 3$$

$$\nu_1^2 = \frac{(P_1|P_1)}{(P_0|P_0)} = \frac{\int_0^\infty (x-1)^2 e^{-x} dx}{\int_0^\infty e^{-x} dx} = 1$$

$$P_2(x) = (x-\mu_1)P_1(x) + \nu_1^2 P_0(x)$$
$$= x^2 - 4x + 2$$

Set $x^2 - 4x + 2 = 0 \Rightarrow x_1 = 2+\sqrt{2},\ x_2 = 2-\sqrt{2}$

$Q(1) = w_1 + w_2 = 1$

$Q(x) = (2+\sqrt{2})w_1 + (2-\sqrt{2})w_2 = 1$

$\Rightarrow \quad w_1 = \frac{1}{4}(2-\sqrt{2})$, $w_2 = \frac{1}{4}(2+\sqrt{2})$

$$Q(x) \approx \sum_{i=1}^{2} w_i f(x_i)$$

$$|E_2(f)| \le \frac{1}{(2\times 2)!} \|f^{(4)}\|_\infty \int_0^\infty (x^2-4x+2)^2 e^{-x} dx$$
$$= \frac{1}{24} \cdot 4 \|f^{(4)}\|_\infty = \frac{1}{6}\|f^{(4)}\|_\infty$$

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5. One - pt Gaussian

$P_0 = 1$

$P_1 = x - \mu_0, \quad \mu_0 = \dfrac{\int_{-1}^{1} x^5 dx}{\int_{-1}^{1} x^4 dx} = 0$

$P_1(x) = x$

$Q_1(f) = f(0)\, w_1, \quad Q_1(1) = w_1 = \frac{2}{5} = I_1$

So, $Q_1(f) = \frac{2}{5} f(0)$

$|E_1| \leq \dfrac{1}{(2n)!} \|f^{(2n)}\|_\infty \int_{-1}^{1} (P_1(x))^2 x^4 dx$

$= \frac{1}{7} \|f^{(2)}\|_\infty$

Similarly,

Two - pts Gaussian

$Q_2(f) = \frac{1}{5} f(\sqrt{\frac{5}{7}}) + \frac{1}{5} f(-\frac{\sqrt{5}}{7})$

$|E_2| \leq \dfrac{1}{27.49} \|f^{(4)}\|_\infty$

Three - Pts Gaussian

$Q_3(f) = \frac{8}{245} f(0) + \frac{18}{98} (f(-\sqrt{\frac{7}{9}}) + f(\sqrt{\frac{7}{9}}))$

$|E_3| \leq \dfrac{\|f^{(6)}\|_\infty}{80190}$

4 - pts Gaussian

$P_4(x) = x^4 - \frac{14}{11}x^2 + \frac{35}{99}$

roots $(\pm 0.93, \pm 0.64)$

$Q_4(x) = 0.076\,[f(0.93) + f(-0.93)] + 0.124\,[f(0.64) + f(-0.64$

$|E_4| \le \|f^{(8)}\|_\infty \cdot (2.492 \cdot 10^{-8})$

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