Nonhomogeneous Linear Equations: Variation of Parameters

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We now return to the discussion of the general case

$$\mathcal{L}(t)y = a_0(t)y'' + a_1(t)y' + a_2(t)y = b(t)\,. \tag{1.1}$$

Important Fact: If you know the general solution of the associated homogeneous problem $\mathcal{L}(t)y = 0$ then you can *always* reduce the construction of the general solution of (1.1) to the problem of finding two primitives (antiderivatives). The method for doing this is called *variation of parameters*.

Because at this point you only know how to find general solutions of homogeneous problems with constant coefficients, problems you will be given will generally fall into one of two categories. Either (1) the problem will have variable coefficients and you will be given a fundamental set of solutions for the associated homogeneous problem, or (2) the problem will have constant coefficients and you will be expected to find a fundamental set of solutions for the associated homogeneous problem.

We shall first illustrate the method of variation of parameters on second order equations in the normal form

$$\mathcal{L}(t)y = y'' + p(t)y' + q(t)y = g(t)\,. \tag{1.2}$$

You can put the general equation (1.1) into normal form by simply dividing by $a_0(t)$.

Suppose you know that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions of the homogeneous problem $\mathcal{L}(t)y = 0$ associated with (1.2). The general solution of the homogeneous problem is then given by

$$y = Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,. \tag{1.3}$$

The idea of the method of variation of parameters is to seek solutions of (1.2) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)\,. \tag{1.4}$$

In other words you simply replace the arbitrary constants c_1 and c_2 in (1.3) with unknown functions $u_1(t)$ and $u_2(t)$. These functions are the varying parameters referred to in the title of the method. These two functions will be governed by a system of two equations, one of which is derived by requiring that (1.2) is satisfied, and the other of which is chosen to simplify the resulting system.

Let us see how this is done. Differentiating (1.4) yields

$$y' = u_1(t)Y_1'(t) + u_2(t)Y_2'(t) + u_1'(t)Y_1(t) + u_2'(t)Y_2(t)\,. \tag{1.5}$$

We now choose to impose the condition

$$u_1'(t)Y_1(t) + u_2'(t)Y_2(t) = 0\,, \tag{1.6}$$

whereby (1.5) simplifies to

$$y' = u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\,. \tag{1.7}$$

Differentiating (1.7) then yields

$$y'' = u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,. \tag{1.8}$$

Now substituting (1.4), (1.7), and (1.8) into (1.2) and using the fact that $Y_1(t)$ and $Y_2(t)$ are solutions of $\mathcal{L}(t)y = 0$ we find that

$$\begin{aligned}
g(t) &= \mathcal{L}(t)y \\
&= y'' + p(t)y' + q(t)y \\
&= u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) \\
&\quad + p(t)u_1(t)Y_1'(t) + p(t)u_2(t)Y_2'(t) \\
&\quad + q(t)u_1(t)Y_1(t) + q(t)u_2(t)Y_2(t) \\
&= u_1(t)\big[Y_1''(t) + p(t)Y_1'(t) + q(t)Y_1(t)\big] \\
&\quad + u_2(t)\big[Y_2''(t) + p(t)Y_2'(t) + q(t)Y_2(t)\big] \\
&\quad + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) \\
&= u_1(t)\big[\mathcal{L}(t)Y_1(t)\big] + u_2(t)\big[\mathcal{L}(t)Y_2(t)\big] \\
&\quad + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) \\
&= u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,.
\end{aligned} \tag{1.9}$$

Here we have used the fact that $\mathcal{L}(t)Y_1(t) = 0$ and $\mathcal{L}(t)Y_2(t) = 0$ to see that many terms in the expression for $\mathcal{L}(t)y$ cancel. The resulting system that governs $u_1(t)$ and $u_2(t)$ is thereby given by (1.6) and (1.9):

$$\begin{aligned}
u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0\,, \\
u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= g(t)\,.
\end{aligned} \tag{1.10}$$

This is a linear system of two algebraic equations for $u_1'(t)$ and $u_2'(t)$. Because

$$\big(Y_1(t)Y_2'(t) - Y_2(t)Y_1'(t)\big) = W(Y_1, Y_2)(t) \neq 0\,,$$

one can always solve this system to find

$$u_1'(t) = -\frac{Y_2(t)g(t)}{W(Y_1, Y_2)(t)}\,, \qquad u_2'(t) = \frac{Y_1(t)g(t)}{W(Y_1, Y_2)(t)}\,.$$

Letting $U_1(t)$ and $U_2(t)$ be any primitives of the respective right-hand sides above, one sees that

$$u_1(t) = c_1 + U_1(t)\,, \qquad u_2(t) = c_2 + U_2(t)\,,$$

whereby (1.4) yields the general solution

$$y = c_1 Y_1(t) + U_1(t)Y_1(t) + c_2 Y_2(t) + U_2(t)Y_2(t)\,.$$

Notice that this decomposes as $y = Y_H(t) + Y_P(t)$ where

$$Y_H(t) = c_1 Y_1(t) + c_2 Y_2(t)\,, \qquad Y_P(t) = U_1(t)Y_1(t) + U_2(t)Y_2(t)\,.$$

The best way to apply this method in practice is not to memorize one of the various formulas for the final solution given in the book, but rather to construct the linear system (1.10), which can then be rather easily solved for $u_1'(t)$ and $u_2'(t)$. Given $Y_1(t)$ and $Y_2(t)$, a fundamental set of solutions to the associated homogeneous problem, you proceed as follows.

1) Write the equation in the normal form

$$y'' + p(t)y' + q(t)y = g(t)\,.$$

2) Write the form of the solution you seek:

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)\,.$$

3) Write the algebraic linear system for $u_1'(t)$ and $u_2'(t)$:

$$\begin{aligned} u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0\,, \\ u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= g(t)\,. \end{aligned}$$

The form of the left-hand sides of this system mimics the form of the solution we seek. The first equation simply replaces $u_1(t)$ and $u_2(t)$ with $u_1'(t)$ and $u_2'(t)$, while the second also replaces $Y_1(t)$ and $Y_2(t)$ with $Y_1'(t)$ and $Y_2'(t)$.

4) Solve the algebraic system for $u_1'(t)$ and $u_2'(t)$. This is always very easy to do, especially if you start with the first equation.

5) Integrate to find $u_1(t)$ and $u_2(t)$. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. Remember to include the constants of integration, c_1 and c_2.

6) Substitute the result into the form of the solution you wrote down in step 2. If the problem is an initial-value problem you must determine c_1 and c_2 from the initial conditions.

Example: Find the general solution of

$$y'' + y = \sec(t)\,.$$

Before presenting the solution, notice that while this equation has constant coefficients, the driving is not of the form that would allow you to use the method of undetermined coefficients. You should be able to recognize this right away and thereby see that the only method you can use to solve this problem is variation of parameters.

The equation is in normal form. Because this problem has constant coefficients, it is easily found that

$$Y_H(t) = c_1 \cos(t) + c_2 \sin(t) .$$

Hence, we will seek a solution of the form

$$y = u_1(t) \cos(t) + u_2(t) \sin(t) ,$$

where

$$\begin{aligned} u_1'(t) \cos(t) + u_2'(t) \sin(t) &= 0 , \\ -u_1'(t) \sin(t) + u_2'(t) \cos(t) &= \sec(t) . \end{aligned}$$

Solving this system by any means you choose yields

$$u_1'(t) = -\frac{\sin(t)}{\cos(t)} , \qquad u_2'(t) = 1 .$$

These can be integrated analytically to obtain

$$u_1(t) = c_1 + \ln(|\cos(t)|) , \qquad u_2(t) = c_2 + t .$$

Therefore the general solution is

$$y = c_1 \cos(t) + c_2 \sin(t) + \ln(|\cos(t)|) \cos(t) + t \sin(t) .$$

Example: Find the general solution of

$$y'' - 2y' + y = \frac{e^t}{1 + t^2} .$$

Before presenting the solution, notice that while this equation has constant coefficients, the driving is not of the form that would allow you to use the method of undetermined coefficients. You thereby see that the only method you can use to solve this problem is variation of parameters.

The equation is in normal form. Because this problem has constant coefficients, it is easily found that

$$Y_H(t) = c_1 e^t + c_2 t e^t .$$

Hence, we will seek a solution of the form

$$y = u_1(t) e^t + u_2(t) t e^t ,$$

where

$$u_1'(t)e^t + u_2'(t)te^t = 0\,,$$

$$u_1'(t)e^t + u_2'(t)(1+t)e^t = \frac{e^t}{1+t^2}\,.$$

Upon factoring out the e^t from each equation, one find that

$$u_1'(t) + u_2'(t)t = 0\,,$$

$$u_1'(t) + u_2'(t)(1+t) = \frac{1}{1+t^2}\,.$$

Solving this system by any means you choose yields

$$u_1'(t) = -\frac{t}{1+t^2}\,, \qquad u_2'(t) = \frac{1}{1+t^2}\,.$$

These can be integrated analytically to obtain

$$u_1(t) = c_1 - \tfrac{1}{2}\ln(1+t^2)\,, \qquad u_2(t) = c_2 + \tan^{-1}(t)\,.$$

Therefore the general solution is

$$y = c_1 e^t + c_2 te^t - \tfrac{1}{2}\ln(1+t^2)e^t + \tan^{-1}(t)te^t\,.$$