Advanced Calculus: MATH 410
Notes on Integrals and Integrability

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1. Definite Integrals

In this section we revisit the definite integral that you were introduced to when you first studied calculus. You undoubtedly learned that given a positive function f over an interval $[a, b]$ the definite integral

$$\int_a^b f(x)\,\mathrm{d}x$$

provided it was defined, was a number equal to the area under the graph of f over $[a, b]$. You also likely learned that the definite integral was defined as a limit of Riemann sums. The Riemann sums you most likely used involved partitioning $[a, b]$ into n uniform subintervals of length $(b - a)/n$ and evaluating f at either the right-hand endpoint, the left-hand endpoint, or the midpoint of each subinterval. At the time your understanding of the notion of limit was likely more intuitive than rigorous. In this section we present a rigorous development of the definite integral built upon the rigorous understanding of limit that you have studied earlier in this course.

1.1. **Partitions and Darboux Sums.**

In the approach taken here, we will consider very general partitions of the interval $[a, b]$, not just those with uniform subintervals.

Definition 1.1. *Let* $[a, b] \subset \mathbb{R}$. *A* partition *of the interval* $[a, b]$ *is specified by* $n \in \mathbb{N}$, *and* $\{x_i\}_{i=1}^n \subset [a, b]$ *such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\,.$$

This partition is denoted $P = [x_0, x_1, \cdots, x_{n-1}, x_n]$. *Each* x_i *for* $i = 0, \cdots, n$ *is called a* partition point of P, *and for each* $i = 1, \cdots, n$ *the interval* $[x_{i-1}, x_i]$ *is called a* i^{th} subinterval induced by P. *The* partition thickness, *denoted* $|P|$, *is defined by*

$$|P| = \max\left\{x_i - x_{i-1} \,:\, i = 1, \cdots, n\right\}.$$

The approach taken here is not based on Riemann sums, but rather on Darboux sums. This is because Darboux sums are well-suited for analysis by the tools we have developed to establish the existence of limits. We will be able to recover results about Riemann sums because, as we will show, every Riemann sum is bounded by two Darboux sums.

Let $f : [a, b] \to \mathbb{R}$ be bounded. Set

$$m = \inf \left\{ f(x) \, : \, x \in [a, b] \right\} , \qquad M = \sup \left\{ f(x) \, : \, x \in [a, b] \right\} . \tag{1}$$

Because f is bounded, one knows that $-\infty < m \le M < \infty$.

Let $P = [x_0, \cdots , x_n]$ be a partition of $[a, b]$. For each $i = 1, \cdots , n$ set

$$\begin{aligned} m_i &= \inf \left\{ f(x) \, : \, x \in [x_{i-1}, x_i] \right\} , \\ M_i &= \sup \left\{ f(x) \, : \, x \in [x_{i-1}, x_i] \right\} . \end{aligned}$$

Clearly $m \le m_i \le M_i \le M$.

Definition 1.2. *The* lower and upper Darboux sums *associated with the function f and partition P are respectively defined by*

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i \, (x_i - x_{i-1}) \, , \\ U(f, P) &= \sum_{i=1}^{n} M_i \, (x_i - x_{i-1}) \, . \end{aligned} \tag{2}$$

Clearly, the Darboux sums satisfy the bounds

$$m \, (b - a) \le L(f, P) \le U(f, P) \le M \, (b - a) \, . \tag{3}$$

These inequalities will all be equalities when f is a constant.

Remark. A Riemann sum associated with the partition P is specified by selecting a quadrature point $q_i \in [x_{i-1}, x_i]$ for each $i = 1, \cdots , n$. Let $Q = (q_1, \cdots , q_n)$ be the n-tuple of quadrature points. The associated Riemann sum is then

$$R(f, P, Q) = \sum_{i=1}^{n} f(q_i) \, (x_i - x_{i-1}) \, . \tag{4}$$

It is easy to see that for any choice of quadrature points Q one has the bounds

$$L(f, P) \le R(f, P, Q) \le U(f, P) \, . \tag{5}$$

Moreover, one can show that

$$\begin{aligned} L(f, P) &= \inf \left\{ R(f, P, Q) \, : \, Q \text{ are quadrature points for } P \right\} , \\ U(f, P) &= \sup \left\{ R(f, P, Q) \, : \, Q \text{ are quadrature points for } P \right\} . \end{aligned}$$

The bounds (5) are thereby sharp.

1.2. **Refinements.** We now introduce the notion of a refinement of a partition.

Definition 1.3. *Given a partition P of an interval $[a,b]$, a partition P^* of $[a,b]$ is called a* refinement *of P provided every partition point of P is a partition point of P^*.*

If $P = [x_0, x_1, \cdots, x_{n-1}, x_n]$ and P^* is a refinement of P then P^* induces a partition of each $[x_{i-1}, x_i]$, which we denote by P_i^*. For example, if $P^* = [x_0^*, x_1^*, \cdots, x_{n^*-1}^*, x_{n^*}^*]$ with $x_{j_i}^* = x_i$ for each $i = 0, \cdots, n$ then $P_i^* = [x_{j_{i-1}}^*, \cdots, x_{j_i}^*]$. Observe that

$$L(f, P^*) = \sum_{i=1}^{n} L(f, P_i^*)\,, \qquad U(f, P^*) = \sum_{i=1}^{n} U(f, P_i^*)\,. \tag{6}$$

Moreover, upon applying the bounds (3) to P_i^* for each $i = 1, \cdots, n$, we obtain the bounds

$$m_i\,(x_i - x_{i-1}) \le L(f, P_i^*) \le U(f, P_i^*) \le M_i\,(x_i - x_{i-1})\,. \tag{7}$$

This observation is key to the proof of the following.

Lemma 1.1. Refinement Lemma. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Let P be a partition of $[a,b]$ and P^* be a refinement of P. Then*

$$L(f, P) \le L(f, P^*) \le U(f, P^*) \le U(f, P)\,. \tag{8}$$

Proof. It follows from (2), (7), and (6) that

$$\begin{aligned}
L(f, P) &= \sum_{i=1}^{n} m_i\,(x_i - x_{i-1}) \\
&\le \sum_{i=1}^{n} L(f, P_i^*) = L(f, P^*) \\
&\le U(f, P^*) = \sum_{i=1}^{n} U(f, P_i^*) \\
&\le \sum_{i=1}^{n} M_i\,(x_i - x_{i-1}) = U(f, P)\,.
\end{aligned}$$

□

1.3. **Comparisons.** A key step in our development will be to develop comparisons of $L(f, P^1)$ and $U(f, P^2)$ for any two partitions P^1 and P^2, of $[a,b]$.

Definition 1.4. *Given two partitions, P^1 and P^2, of $[a,b]$ define $P^1 \vee P^2$ to be the partition whose set of partition points is the union of the partition points of P^1 and the partition points of P^2. We call $P^1 \vee P^2$ the* supremum *of P^1 and P^2.*

It is easy to argue that $P^1 \vee P^2$ is the smallest partition of $[a,b]$ that is a refinement of both P^1 and P^2.

Lemma 1.2. Comparison Lemma. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Let P^1 and P^2 be partitions of $[a,b]$. Then*

$$L(f, P^1) \leq U(f, P^2)\,. \tag{9}$$

Proof. Because $P^1 \vee P^2$ is a refinement of both P^1 and P^2, it follows from the Refinement Lemma that

$$L(f, P^1) \leq L(f, P^1 \vee P^2) \leq U(f, P^1 \vee P^2) \leq U(f, P^2)\,.$$

□

Because the partitions P^1 and P^2 on either side of inequality (9) are independent, we may obtain sharper bounds by taking the supremum over P^1 on the right-hand side, or the infimum over P^2 on the left-hand side. Indeed, we prove the following.

Lemma 1.3. Sharp Comparison Lemma. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Let*

$$\begin{aligned} \overline{L}(f) &= \sup\big\{ L(f,P) \,:\, P \textit{ is a partition of } [a,b] \big\}\,, \\ \underline{U}(f) &= \inf\big\{ U(f,P) \,:\, P \textit{ is a partition of } [a,b] \big\}\,. \end{aligned} \tag{10}$$

Let P^1 and P^2 be partitions of $[a,b]$. Then

$$L(f, P^1) \leq \overline{L}(f) \leq \underline{U}(f) \leq U(f, P^2)\,. \tag{11}$$

Remark. Because it is clear from (10) that $\overline{L}(f)$ and $\underline{U}(f)$ depend on $[a,b]$, strictly speaking these quantities should be denoted $\overline{L}(f,[a,b])$ and $\underline{U}(f,[a,b])$. This would be necessary if more than one interval was involved in the discussion. However, that is not the case here. We therefore embrace the less cluttered notation.

Proof. If we take the infimum of the right-hand side of (9) over P^2, we obtain

$$L(f, P_1) \leq \underline{U}(f)\,.$$

If we then take the supremum of the left-hand side above over P^1, we obtain

$$\overline{L}(f) \leq \underline{U}(f)\,.$$

The bound (11) then follows. □

1.4. **Definition of the Definite Integral.** We are now ready to define the definite integral. You will find different definitions of the definite integral in different books. Here we will used the definition found in Fitzpatrick's book. We will then give a theorem that shows this definition is equivalent to another one commonly found in other books.

Definition 1.5. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is said to be* integrable *over $[a,b]$ whenever there exists a unique $A \in \mathbb{R}$ such that*

$$(12) \qquad L(f,P) \leq A \leq U(f,P) \quad \text{for every partition } P \text{ of } [a,b]\,.$$

In this case we call A the definite integral *of f over $[a,b]$ and denote it by $\int_a^b f$.*

Remark. The Sharp Comparison Lemma shows that (12) holds for every $A \in [\overline{L}(f), \underline{U}(f)]$. The key thing to be established when using the above definition is therefore the uniqueness of such an A.

We now give the following characterizations of integrability.

Theorem 1.1. Integrability Theorem. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then the following are equivalent:*

(1) *f is integrable over $[a,b]$;*
(2) *$\overline{L}(f) = \underline{U}(f)$;*
(3) *for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that*

$$0 \leq U(f,P) - L(f,P) < \epsilon\,.$$

Proof. We first show that (1) $\implies$ (2). Suppose that (2) is false. Then $\overline{L}(f) < \underline{U}(f)$. Observe that the Sharp Comparison Lemma shows that for every $A \in [\overline{L}(f), \underline{U}(f)]$ one has

$$L(f,P^1) \leq A \leq U(f,P^2) \quad \text{for any partitions } P^1 \text{ and } P^2 \text{ of } [a,b]\,.$$

Hence, there are many values of A that satisfy (12), whereby f is not integrable. It follows that (1) $\implies$ (2).

Next we show that (2) $\implies$ (3). Let $\epsilon > 0$. By the definition (10) of $\overline{L}(f)$ and $\underline{U}(f)$, we can find partitions P^1 and P^2 of $[a,b]$ such that

$$\overline{L}(f) - \frac{\epsilon}{2} < L(f,P^1) \leq \overline{L}(f)\,,$$
$$\underline{U}(f) \leq U(f,P^2) < \underline{U}(f) + \frac{\epsilon}{2}\,.$$

Now let $P = P^1 \vee P^2$. Because the Comparison Lemma implies that $L(f,P^1) \leq L(f,P)$ and $U(f,P) \leq U(f,P^2)$, it follows from the above inequalities that

$$\overline{L}(f) - \frac{\epsilon}{2} < L(f,P) \leq \overline{L}(f)\,,$$
$$\underline{U}(f) \leq U(f,P) < \underline{U}(f) + \frac{\epsilon}{2}\,.$$

Hence, if $\overline{L}(f) = \underline{U}(f)$ one thereby concludes that

$$0 \leq U(f,P) - L(f,P) < \left(\underline{U}(f) + \frac{\epsilon}{2}\right) - \left(\overline{L}(f) - \frac{\epsilon}{2}\right) = \epsilon\,.$$

This shows that (2) $\Longrightarrow$ (3).

Finally, we show that (3) $\Longrightarrow$ (1). Suppose that (1) is false. Then by the Sharp Comparison Lemma there exists A_1 and A_2 such that

$$L(f,P) \leq A_1 < A_2 \leq U(f,P) \quad \text{for every partition } P \text{ of } [a,b]\,.$$

One thereby has that

$$U(f,P) - L(f,P) \geq A_2 - A_1 \quad \text{for every partition } P \text{ of } [a,b]\,.$$

Hence, (3) must be false. It follows that (1) $\Longrightarrow$ (2). □

Remark. Property (3) of the Integrability Theorem provides a very useful criterion for establishing the integrability of a function f. We will exploit this in the next section.

Remark. It follows from the Integrability Theorem that one could well adopt the following alternative definition of the definite integral.

Definition 1.5′. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is said to be* integrable *over $[a,b]$ whenever $\overline{L}(f) = \underline{U}(f)$. In this case we call this common value the* definite integral *of f over $[a,b]$ and denote it by $\int_a^b f$.*

This is a definition of the definite integral that is commonly found in textbooks.

2. Integrable Functions and Integrals

In this section we use the Integrability Theorem to develop criteria to identify integrable functions. We also begin the task of evaluating definite integrals.

2.1. Evaluating Integrals via Riemann Sums.

We now make the connection with the notion of a definite integral as the limit of a sequence of Riemann sums.

Theorem 2.1. Riemann Sums Convergence Theorem. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Let $\{P^n\}_{n=1}^{\infty}$ be a sequence of partitions of $[a,b]$ such that*

$$\lim_{n\to\infty} \big(U(f,P^n) - L(f,P^n) \big) = 0 \,. \tag{13}$$

Let Q^n be any quadrature set associated with P^n. Then f is integrable over $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty} R(f,P^n,Q^n) \,, \tag{14}$$

where the Riemann sums $R(f,P,Q)$ are defined by (4).

Remark. The content of this theorem is that once one has found a sequence of partitions P^n such that (13) holds, then the integral $\int_a^b f$ exists and may be evaluated as the limit of any associated sequence of Riemann sums (14). This theorem thereby splits the task of evaluating a definite integrals into two steps. The first step is by far the easier. It is a rare integrand f for which one can find a sequence of Riemann that allow one to evaluate the limit in (14).

Proof. Given (13), the fact that f is integrable over $[a,b]$ follows directly from criterion (3) of the Integrability Theorem.

The bounds on Riemann sums given by (5) yield the inequalities

$$L(f,P^n) \le R(f,P^n,Q^n) \le U(f,P^n) \,,$$

while, because f is integrable, we also have the inequalities

$$L(f,P^n) \le \int_a^b f \le U(f,P^n) \,.$$

It follows from these inequalities that

$$\begin{aligned} L(f,P^n) - U(f,P^n) &\le L(f,P^n) - \int_a^b f \\ &\le R(f,P^n,Q^n) - \int_a^b f \\ &\le U(f,P^n) - \int_a^b f \le U(f,P^n) - L(f,P^n) \,, \end{aligned}$$

which implies that

$$\left| R(f,P^n,Q^n) - \int_a^b f \right| \le U(f,P^n) - L(f,P^n) \,.$$

Because (13) states that the right-hand side above vanishes as n tends to ∞, the limit (14) follows. □

2.2. **Monotone and Piecewise Monotone Functions.** We now use the Riemann Sums Convergence Theorem to show that the class of integrable functions includes the class of monotone functions. Recall that this class is defined as follows.

Definition 2.1. *A function* $f : [a,b] \to \mathbb{R}$ *is said to be* monotonically increasing *provided that*

$$x < y \implies f(x) \leq f(y) \quad \textit{for every } x, y \in [a,b]\,.$$

A function $f : [a,b] \to \mathbb{R}$ *is said to be* monotonically decreasing *provided that*

$$x < y \implies f(x) \geq f(y) \quad \textit{for every } x, y \in [a,b]\,.$$

If a function is either monotonically increasing or monotonically decreasing then it is said to be monotone*.*

It is a classical fact that a monotone function over $[a,b]$ is continuous at all but at most a countable number of points where it has a jump discontinuity.

Theorem 2.2. *Let* $f : [a,b] \to \mathbb{R}$ *be monotone. Then* f *is integrable over* $[a,b]$*.*

Moreover, for any sequences P^n *of partitions of* $[a,b]$ *and* Q^n *of associated quadrature points such that* $|P^n| \to 0$ *as* $n \to \infty$*, one has that*

$$\int_a^b f = \lim_{n\to\infty} R(f, P^n, Q^n)\,,$$

where the Riemann sums $R(f,P,Q)$ *are defined by (4).*

Proof. For any partition $P = [x_0, \cdots, x_n]$ we have the following basic estimate. Because f is monotone, over each subinterval $[x_{i-1}, x_i]$ one has that

$$M_i - m_i = \big|f(x_i) - f(x_{i-1})\big|\,.$$

We thereby obtain the basic estimate

$$\begin{aligned} 0 \leq U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)\,(x_i - x_{i-1}) \\ &\leq |P| \sum_{i=1}^{n} (M_i - m_i) = |P| \sum_{i=1}^{n} \big|f(x_i) - f(x_{i-1})\big| \\ &= |P|\,\big|f(b) - f(a)\big|\,, \end{aligned}$$

where $|P| = \max\{x_i - x_{i-1} \,:\, i = 1, \cdots, n\}$ is the thickness of P. Here we have used the fact that, because f is monotone, the terms $f(x_i) - f(x_{i-1})$ are either all nonnegative, or all nonpositive. This fact allows us to pass the absolute value outside the sum, which then telescopes.

We now apply the above basic estimate to our sequence P^n of partitions, which shows that

$$0 \leq U(f, P^n) - L(f, P^n) \leq |P^n| \, \big|f(b) - f(a)\big| \to 0 \quad \text{as } n \to \infty \, .$$

The result then follows from the Riemann Sums Convergence Theorem. □

Example. One can use this theorem to show that for every $k \in \mathbb{N}$ the function $x \mapsto x^k$ is integrable over $[0, b]$ and that

$$\int_0^b x^k \, \mathrm{d}x = \lim_{n \to \infty} \left[\frac{b^{k+1}}{n^{k+1}} \sum_{i=1}^{n} i^k \right] = \frac{b^{k+1}}{k+1} \, . \tag{15}$$

The details of this calculation are presented in the book for the cases $k = 0, 1, 2$ with $b = 1$. Here we present the general case. Define

$$S^k(n) = \sum_{i=1}^{n} i^k \, .$$

In order to prove (15) we must establish the limit

$$\lim_{n \to \infty} \frac{1}{n^{k+1}} \, S^k(n) = \frac{1}{k+1} \, . \tag{16}$$

We do this below by induction on k.

Proof. Clearly $S^0(n) = n$, so that limit (16) holds for $k = 0$. Now assume that for some $l \geq 1$ limit (16) holds for every $k < l$. By a telescoping sum, a binomial expansion, and the definition of $S^k(n)$, one obtains the identity

$$\begin{aligned}
(n+1)^{l+1} - 1 &= \sum_{i=1}^{n} \left[(i+1)^{l+1} - i^{l+1} \right] \\
&= \sum_{i=1}^{n} \sum_{j=0}^{l} \frac{(l+1)!}{j!(l-j+1)!} \, i^j \\
&= \sum_{j=0}^{l} \frac{(l+1)!}{j!(l-j+1)!} \, S^j(n) \\
&= (l+1) \, S^l(n) + \sum_{j=0}^{l-1} \frac{(l+1)!}{j!(l-j+1)!} \, S^j(n) \, .
\end{aligned}$$

Upon solving for $S^l(n)$ and dividing by n^{l+1}, we obtain the relation

$$
\begin{aligned}
\frac{1}{n^{l+1}}\, S^l(n) = \frac{1}{l+1}\Bigg[&\frac{(n+1)^{l+1}}{n^{l+1}} - \frac{1}{n^{l+1}} \\
&- \sum_{j=0}^{l-1} \frac{(l+1)!}{j!(l-j+1)!}\,\frac{1}{n^{l+1}}\, S^j(n)\Bigg] .
\end{aligned}
\tag{17}
$$

Because we know

$$
\lim_{n\to\infty} \frac{(n+1)^{l+1}}{n^{l+1}} = 1\,, \qquad \lim_{n\to\infty} \frac{1}{n^{l+1}} = 0\,,
$$

and because, by the induction hypothesis, we know

$$
\lim_{n\to\infty} \frac{1}{n^{l+1}}\, S^j(n) = 0 \quad \text{for every } j < l\,,
$$

we can pass to the $n \to \infty$ limit in relation (17). We thereby establish that limit (16) holds for $k = l$. □

We now use the Riemann Sums Convergence Theorem to show that the class of integrable functions includes the class of piecewise monotone functions. Recall that this class is defined as follows.

Definition 2.2. *A function $f : [a,b] \to \mathbb{R}$ is said to be* piecewise monotone *over $[a,b]$ provided there exists a partition $[p_0, \cdots, p_m]$ of $[a,b]$ such that f is monotone over $[p_{j-1}, p_j]$ for every $j = 1, \cdots, m$.*

Theorem 2.3. Piecewise Monotone Integrability Theorem. *Let $f : [a,b] \to \mathbb{R}$ be piecewise monotone over $[a,b]$. Then f is integrable over $[a,b]$.*

Moreover, for any sequences P^n of partitions of $[a,b]$ and Q^n of associated quadrature points such that $|P^n| \to 0$ as $n \to \infty$, one has that

$$
\int_a^b f = \lim_{n\to\infty} R(f, P^n, Q^n)\,,
$$

where the Riemann sums $R(f,P,Q)$ are defined by (4).

Proof. Let $[p_0, \cdots, p_m]$ be a partition of $[a,b]$ such that f is monotone over $[p_{j-1}, p_j]$ for each $j = 1, \cdots, m$. Let $P = [x_0, \cdots, x_n]$ be any partition of $[a,b]$. The key step will be to establish the bound

$$
\sum_{i=1}^{n} (M_i - m_i) \le \sum_{j=1}^{m} \big| f(p_j) - f(p_{j-1}) \big| . \tag{18}
$$

Once this is done we can obtain the basic estimate

$$
\begin{aligned}
0 \le U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)\,(x_i - x_{i-1}) \\
&\le |P| \sum_{i=1}^{n} (M_i - m_{i-1}) \\
&\le |P| \sum_{i=1}^{m} \bigl| f(p_j) - f(p_{j-1}) \bigr| \,,
\end{aligned}
$$

where $|P| = \max\{x_i - x_{i-1} \,:\, i = 1, \cdots, n\}$ is the thickness of P.

By then applying the above basic estimate to our sequence P^n of partitions, we see that

$$
\begin{aligned}
0 &\le U(f,P^n) - L(f,P^n) \\
&\le |P^n| \sum_{i=1}^{m} \bigl| f(p_j) - f(p_{j-1}) \bigr| \to 0 \quad \text{as } n \to \infty \,.
\end{aligned}
$$

The result would then follows from the Riemann Sums Convergence Theorem.

All that remains to be done is establish the bound (18). This is easy to do when P is a refinement of $[p_0, \cdots, p_m]$. When P is not a refinement of $[p_0, \cdots, p_m]$ one simply replaces P with $P \vee [p_0, \cdots, p_m]$. We leave the details of these arguments as an exercise. □

2.3. **Piecewise Integrability.** A key tool for building up the class of integrable functions is the the following lemma.

Lemma 2.1. Piecewise Integrability Lemma. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Let $P = [p_0, \cdots, p_k]$ be a partition of $[a,b]$ such that f is integrable over $[p_{i-1}, p_i]$ for every $i = 1, \cdots, k$. Then f is integrable over $[a,b]$. Moreover,*

$$
\int_a^b f = \sum_{i=1}^{k} \int_{p_{i-1}}^{p_i} f \,. \tag{19}
$$

Proof. Let $\epsilon > 0$. Because f is integrable over $[p_{i-1}, p_i]$ there exists a partition P_i^* of $[p_{i-1}, p_i]$ such that

$$
0 \le U(f, P_i^*) - L(f, P_i^*) < \frac{\epsilon}{k} \,.
$$

Let P^* be the refinement of P such that P_i^* is the induced partition of $[p_{i-1}, p_i]$. One then sees that

$$\begin{aligned} 0 &\leq U(f, P^*) - L(f, P^*) \\ &= \sum_{i=1}^{k} \big(U(f, P_i^*) - L(f, P_i^*)\big) < \sum_{i=1}^{k} \frac{\epsilon}{k} = \epsilon \, . \end{aligned}$$

Hence, by item (3) of the Integrability Theorem, f is integrable.

Let P^* be a partition of $[a, b]$. Without loss of generality we may assume that P^* is a refinement of P, otherwise pass to the partition $P^* \vee P$. Let P_i^* be the induced partition of $[p_{i-1}, p_i]$. Because f is integrable over $[p_{i-1}, p_i]$ we have that

$$L(f, P_i^*) \leq \int_{p_{i-1}}^{p_i} f \leq U(f, P_i^*) \, .$$

Summing these inequalities over $i = 1, \cdots, k$ yields

$$L(f, P^*) \leq \sum_{i=1}^{k} \int_{p_{i-1}}^{p_i} f \leq U(f, P^*) \, .$$

Formula (19) then follows. □

2.4. **Piecewise Continuous Functions.** We now show that all functions that are piecewise continuous over $[a, b]$ are also intergrable over $[a, b]$. We first recall the definition of piecewise continuous function.

Definition 2.3. *A function $f : [a, b] \to \mathbb{R}$ is said to be* piecewise continuous *if it is bounded and there exists a partition $P = [x_0, \cdots, x_n]$ of $[a, b]$ such that f is continuous over (x_{i-1}, x_i) for every $i = 1, \cdots, n$.*

We remark that piecewise continuous functions are discontinuous at only a finite number of points. Still, the class of piecewise continuous functions includes some fairly wild functions. For example, it contains the function

$$f(x) = \begin{cases} 1 + \sin(1/x) & \text{if } x > 0 \, , \\ 4 & \text{if } x = 0 \, , \\ -1 + \sin(1/x) & \text{if } x < 0 \, , \end{cases}$$

considered over $[-1, 1]$. As wild as this function looks, it is continuous everywhere except at the point $x = 0$.

We will need two lemmas.

Lemma 2.2. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable over $[a, b]$.*

Proof. Let $\epsilon > 0$. Because f is uniformly continuous over $[a, b]$, there exists a $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b-a} \quad \text{for every } x, y \in [a, b] \,.$$

Pick $n \in \mathbb{N}$ such that

$$\frac{b-a}{n} < \delta \,.$$

Let $P = [x_0, \cdots, x_n]$ be the partition of $[a, b]$ with

$$x_i = a + i\,\frac{b-a}{n} \quad \text{for every } i = 0, \cdots, n \,.$$

For every $i = 1, \cdots, n$ over the subinterval $[x_{i-1}, x_i]$ one has

$$x_i - x_{i-1} = \frac{b-a}{n} \,, \qquad M_i - m_i \le \frac{\epsilon}{b-a} \,.$$

One thereby sees that

$$\begin{aligned} 0 \le U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &= \frac{b-a}{n} \sum_{i=1}^{n} (M_i - m_i) \\ &< \frac{b-a}{n} \sum_{i=1}^{n} \frac{\epsilon}{b-a} = \frac{b-a}{n}\,\frac{n\,\epsilon}{b-a} = \epsilon \,. \end{aligned}$$

Hence, by item (3) of the Integrability Theorem, f is integrable. □

Lemma 2.3. *Let $f : [a, b] \to \mathbb{R}$ be bounded and $f : (a, b) \to \mathbb{R}$ be continuous. Then f is integrable over $[a, b]$.*

Proof. Let $\epsilon > 0$. Let $\delta > 0$ such that

$$\delta < \frac{\epsilon}{3(M - m)} \,, \qquad \delta < \frac{b-a}{2} \,.$$

where, as before, m and M are defined by (1).

Because f is continuous over $[a + \delta, b - \delta]$, by the previous lemma it is integrable over $[a + \delta, b - \delta]$. By the Integrability Theorem, there exists a partition P^δ of $[a + \delta, b - \delta]$ such that

$$0 \le U(f, P^\delta) - L(f, P^\delta) < \frac{\epsilon}{3} \,.$$

Let P be the partition of $[a, b]$ given by $P = [x_0, x_1, \cdots, x_{n-1}, x_n]$ where $P^\delta = [x_1, \cdots, x_{n-1}]$. One has

$$x_1 - x_0 = x_n - x_{n-1} = \delta \,, \quad M_1 - m_1 < M - m \,, \quad M_n - m_n < M - m \,.$$

One thereby sees that

$$\begin{aligned}0 \le U(f,P) - L(f,P) &= \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) \\ &= (M_1 - m_1)\delta + (M_n - m_n)\delta + \sum_{i=2}^{n-1}(M_i - m_i)(x_i - x_{i-1}) \\ &= (M_1 - m_1)\delta + (M_n - m_n)\delta + U(f,P^\delta) - L(f,P^\delta) \\ &< 2\,(M-m)\,\frac{\epsilon}{3(M-m)} + \frac{\epsilon}{3} = \epsilon\,.\end{aligned}$$

Hence, by item (3) of the Integrability Theorem, f is integrable. □

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be piecewise continuous. Then f is integrable over $[a,b]$.*

Proof. The theorem follows from the previous lemma and the Piecewise Integrability Lemma. □

3. The First Fundamental Theorem of Calculus

The business of evaluating integrals by taking limits of Riemann sums is usually either difficult or impossible. However, as you have known since you first studied integration, for many integrands there is a must easier way.

Theorem 3.1. First Fundamental Theorem of Calculus. *Let $f : [a,b] \to \mathbb{R}$ be integrable. Suppose that $F : [a,b] \to \mathbb{R}$ is continuous, that $F : (a,b) \to \mathbb{R}$ is differentiable, and that*

$$F'(x) = f(x) \quad \text{for every } x \in (a,b)\,. \tag{20}$$

Then

$$\int_a^b f = F(b) - F(a)\,.$$

Remark. This theorem essentially reduces the problem of evaluating definite integrals to that of finding an explicit solution of the differential equation (20). While such an explicit solution cannot always be found, for a wide class of integrands f.

Proof. We must show that for every partition P of $[a,b]$ one has

$$L(f,P) \le F(b) - F(a) \le U(f,P)\,. \tag{21}$$

Let $P = [x_0, \cdots, x_n]$ be an arbitrary partition of $[a,b]$. For every $i = 1, \cdots, n$ one knows that $F : [x_{i-1}, x_i] \to \mathbb{R}$ is continuous, and that

$F : (x_{i-1}, x_i) \to \mathbb{R}$ is differentiable. Then by the Lagrange Mean Value Theorem there exists $q_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(q_i)\,(x_i - x_{i-1}) = f(q_i)\,(x_i - x_{i-1})\,.$$

Because $m_i \leq f(q_i) \leq M_i$, we see from the above that

$$m_i\,(x_i - x_{i-1}) \leq F(x_i) - F(x_{i-1}) \leq M_i\,(x_i - x_{i-1})\,.$$

Finally, adding these inequalities yields (21). □

The following is an immediate corollary of the First Fundamental Theorem of Calculus.

Corollary 3.1. *Let $F : [a, b] \to \mathbb{R}$ be continuous, $F : (a, b) \to \mathbb{R}$ be differentiable, and $F' : (a, b) \to \mathbb{R}$ be continuous and bounded. Let f be any extension of F' to $[a, b]$. Then*

$$\int_a^b f = F(b) - F(a)\,.$$

Example. Let F be defined over $[0, 1]$ by

$$F(x) = \begin{cases} x\cos(\log(1/x)) & \text{if } 0 < x \leq 1\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then F is continuous over $[0, 1]$ and differentiable over $(0, 1]$ with

$$F'(x) = \cos(\log(1/x)) + \sin(\log(1/x))\,.$$

As this function is bounded, we have

$$\int_0^1 [\cos(\log(1/x)) + \sin(\log(1/x))]\,\mathrm{d}x = F(1) - F(0) = 0\,.$$

Here the integrand can be assigned any value at $x = 0$.