

Math 246, Sample Problem Solutions for Third In-Class Exam

(1) Consider the matrices

$$A = \begin{pmatrix} 3 & 1-i \\ 2+i & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix}.$$

Compute the matrices

(a) A^T ,

Solution: The transpose of A is given by

$$A^T = \begin{pmatrix} 3 & 2+i \\ 1-i & 4 \end{pmatrix}.$$

(b) \overline{A} ,

Solution: The conjugate of A is given by

$$\overline{A} = \begin{pmatrix} 3 & 1+i \\ 2-i & 4 \end{pmatrix}.$$

(c) A^* ,

Solution: The adjoint of A is given by

$$A^* = \begin{pmatrix} 3 & 2-i \\ 1+i & 4 \end{pmatrix}.$$

(d) $2A - B$,

Solution: The difference of $2A$ and B is given by

$$2A - B = \begin{pmatrix} 6 & 2-i2 \\ 4+i2 & 8 \end{pmatrix} - \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 0 & -5-i2 \\ -1+i2 & 2 \end{pmatrix}.$$

(e) AB ,

Solution: The product of A and B is given by

$$\begin{aligned} AB &= \begin{pmatrix} 3 & 1-i \\ 2+i & 4 \end{pmatrix} \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 3 \cdot 6 + (1-i) \cdot 5 & 3 \cdot 7 + (1-i) \cdot 6 \\ (2+i) \cdot 6 + 4 \cdot 5 & (2+i) \cdot 7 + 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 23-i5 & 27-i6 \\ 32+i6 & 38+i7 \end{pmatrix}. \end{aligned}$$

(f) B^{-1} .

Solution: Observe that it is clear that B has an inverse because

$$\det(B) = \det \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} = 6 \cdot 6 - 5 \cdot 7 = 36 - 35 = 1 \neq 0.$$

The inverse of B is given by

$$B^{-1} = \begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix}.$$

This may be computed in a number of ways. Here are three.

First, it can be computed by applying elementary row operations to transform the augmented matrix $(B|I)$ into $(I|B^{-1})$ as follows:

$$\begin{aligned} (B \mid I) &= \left(\begin{array}{cc|cc} 6 & 7 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & \frac{7}{6} & \frac{1}{6} & 0 \\ 5 & 6 & 0 & 1 \end{array} \right) \\ &\sim \left(\begin{array}{cc|cc} 1 & \frac{7}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & -\frac{5}{6} & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & \frac{7}{6} & \frac{1}{6} & 0 \\ 0 & 1 & -5 & 6 \end{array} \right) \\ &\sim \left(\begin{array}{cc|cc} 1 & 0 & 6 & -7 \\ 0 & 1 & -5 & 6 \end{array} \right) = (I \mid B^{-1}) . \end{aligned}$$

Second, because B is a two-by-two matrix, its inverse can be computed directly from the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{whenever } ad-bc \neq 0 .$$

This formula is just formula (24) on page 352 of the book specialized to the two-by-two case. When applied to B it yields

$$B^{-1} = \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix}^{-1} = \frac{1}{36-35} \begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix} .$$

Finally, the inverse of B may be computed directly from its definition by seeking a , b , c , and d such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 6a+7c & 6b+7d \\ 5a+6c & 5b+7d \end{pmatrix} .$$

Then a and c are found by solving the two-by-two system

$$6a + 7c = 1, \quad 5a + 6c = 0,$$

which gives $a = 6$ and $c = -5$. Similarly b and d are found by solving the two-by-two system

$$6b + 7d = 0, \quad 5b + 6d = 1,$$

which gives $b = -7$ and $d = 6$. You thereby find that

$$B^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ -5 & 6 \end{pmatrix} .$$

(2) Consider the matrix

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} .$$

(a) Find all the eigenvalues of A .

Solution: The eigenvalues are the roots of the equation

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 - 3^2 .$$

The eigenvalues are therefore $\lambda = 5 \pm 3$, or simply 2 and 8.

(b) For each eigenvalue of A find an eigenvector.

Solution: A vector v is an eigenvector of A corresponding to an eigenvalue λ provided it is a nonzero solution of $(A - \lambda I)v = 0$. The approach taken in the book is to solve these equations.

An eigenvector of A corresponding to $\lambda = 2$ satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5-2 & -3 \\ -3 & 5-2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This leads to the equation $v_1 = v_2$. An eigenvector of A corresponding to the eigenvalue 2 is thereby

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

or any nonzero multiple of it.

An eigenvector of A corresponding to $\lambda = 8$ satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5-8 & -3 \\ -3 & 5-8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This leads to the equation $v_1 = -v_2$. An eigenvector of A corresponding to the eigenvalue 8 is thereby

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

or any nonzero multiple of it.

The approach taken in class was based on the Cayley-Hamilton theorem, which states that

$$0 = p(A) = (A - 2I)(A - 8I) = (A - 8I)(A - 2I).$$

Hence, any nonzero column of $(A - 8I)$ is an eigenvector of $\lambda = 2$ while any nonzero column of $(A - 2I)$ is an eigenvector of $\lambda = 8$. Because

$$A - 8I = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}, \quad A - 2I = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix},$$

you can read off that an eigenvector of A corresponding to the eigenvalue 2 is

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

or any nonzero multiple of it, while an eigenvector of A corresponding to the eigenvalue 8 is

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

or any nonzero multiple of it.

(3) Consider the linear algebraic system

$$x_1 + 2x_2 - x_3 = 1,$$

$$2x_1 + x_2 + x_3 = 1,$$

$$x_1 - x_2 + 2x_3 = 1.$$

Either find its general solution or else show that it has no solution.

Solution: First, a remark. By any method you can compute

$$\det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} = 0.$$

This does not mean however that there is no solution. All it means is that there *might* be no solution. Whether there is a solution or not depends on the right-hand side of the system. For example, if the numbers on the right-hand side are 1, 2, and 1 rather than 1, 1, and 1, then the system would have the solution $x_1 = 1$, $x_2 = 0$ and $x_3 = 0$.

In fact, the given system has no solution. There are many ways to see this. One of the simplest is to add the first and third equations to obtain

$$2x_1 + x_2 + x_3 = 2,$$

which clearly contradicts the second equation. Done.

You can find similar contradictions by taking other combinations of the equations. For example, subtracting the first from the second contradicts the third. Done.

Alternatively, you can try to solve the system by applying elementary row operations to the augmented matrix. This gives

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & -3 & 3 & 0 \end{array} \right).$$

The last two rows on the right-hand side are clearly contradictory, whereby the system has no solution. Done.

If you did not see the contradiction above and continued to apply elementary row operations, you would get

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & -3 & 3 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{3} \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{array} \right).$$

At this point the last row clearly shows that the system has no solution. You have arrived at the right answer, but it took you too long to get there!

- (4) Solve each of the following initial-value problems.

(a)

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Solution: First, you must compute the characteristic polynomial $p(z)$ of the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Because A is 2×2 this can either be done as

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(A)z + \det(A) \\ &= z^2 - (1 - 2)z + (-2 - 4) = z^2 + z - 6, \end{aligned}$$

or as

$$\begin{aligned} p(z) &= \det(A - zI) \\ &= \det \begin{pmatrix} 1-z & 1 \\ 4 & -2-z \end{pmatrix} = (1-z)(-2-z) - 4 \\ &= z^2 - (1-2)z - 2 - 4 = z^2 + z - 6. \end{aligned}$$

By either method, after factoring you obtain

$$p(z) = (z+3)(z-2),$$

whereby the eigenvalues of A are -3 and 2 .

Now there are several approaches you can take. The approach used in class goes as follows. First, compute the exponential matrix $\exp(At)$. Because A has two simple real roots one has that

$$\begin{aligned} \exp(At) &= \frac{1}{2 - (-3)} [(2I - A)e^{-3t} + (A + 3I)e^{2t}] \\ &= \frac{1}{5} \left[\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} e^{2t} \right]. \end{aligned}$$

The solution of the initial-value problem is then given by

$$\begin{aligned} x(t) &= \exp(At) \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \frac{1}{5} \left[\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{2t} \right] \\ &= \frac{1}{5} \left[\begin{pmatrix} 3 \\ -12 \end{pmatrix} e^{-3t} + \begin{pmatrix} 7 \\ 7 \end{pmatrix} e^{2t} \right]. \end{aligned}$$

The approach used in the book goes as follows. First compute eigenvectors associated with the eigenvalues -3 and 2 respectively:

$$\begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution is thereby found to be

$$x(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The initial condition then leads to the equations

$$c_1 + c_2 = 2, \quad -4c_1 + c_2 = -1.$$

These are then solved to find $c_1 = 3/5$ and $c_2 = 7/5$. Hence,

$$x(t) = \frac{3}{5} \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + \frac{7}{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Notice that the details of finding the eigenvectors and of solving for c_1 and c_2 are omitted above. When these details are added it should be clear to you that this approach takes much longer than the approach used in class.

(b)

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution: First, you must compute the characteristic polynomial $p(z)$ of the coefficient matrix

$$A = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}.$$

Because A is 2×2 this can either be done as

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(A)z + \det(A) \\ &= z^2 - (-1 - 1)z + (1 - 4) = z^2 + 2z - 3, \end{aligned}$$

or as

$$\begin{aligned} p(z) &= \det(A - zI) \\ &= \det \begin{pmatrix} -1 - z & 4 \\ 1 & -1 - z \end{pmatrix} = (z - 1)^2 - 4. \end{aligned}$$

By either method, after factoring you obtain

$$p(z) = (z + 3)(z - 1),$$

whereby the eigenvalues of A are -3 and 1 .

Now there are several approaches you can take. The approach used in class goes as follows. First, compute the exponential matrix $\exp(At)$. Because A has two simple real roots one has that

$$\begin{aligned} \exp(At) &= \frac{1}{1 - (-3)} [(I - A)e^{-3t} + (A + 3I)e^t] \\ &= \frac{1}{4} \left[\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} e^t \right]. \end{aligned}$$

The solution of the initial-value problem is then given by

$$\begin{aligned} x(t) &= \exp(At) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{4} \left[\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \right] \\ &= \frac{1}{4} \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 6 \\ 3 \end{pmatrix} e^t \right]. \end{aligned}$$

The approach used in the book goes as follows. First compute eigenvectors associated with the eigenvalues -3 and 2 respectively:

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is thereby found to be

$$x(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t.$$

The initial condition then leads to the equations

$$2c_1 + 2c_2 = 1, \quad -c_1 + c_2 = 1.$$

These are then solved to find $c_1 = -1/4$ and $c_2 = 3/4$. Hence,

$$x(t) = \frac{1}{4} \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + \frac{3}{4} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

Notice that the details of finding the eigenvectors and of solving for c_1 and c_2 are omitted above. When these details are added it should be clear to you that this approach takes much longer than the approach used in class.

- (5) Find a general solution for each of the following systems.
(a)

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Solution: The characteristic polynomial of the coefficient matrix A is

$$p(z) = z^2 - 4z + 8 = (z - 2)^2 + 4.$$

This has the roots $z = 2 \pm i2$.

Now there are several approaches you can take. The approach used in class goes as follows. First, compute the exponential matrix $\exp(At)$. Because A has the simple conjugate pair of roots $2 \pm i2$ one has that

$$\begin{aligned} \exp(At) &= I e^{2t} \cos(2t) + (A - 2I) e^{2t} \frac{\sin(2t)}{2} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{2t} \cos(2t) + \begin{pmatrix} -2 & -4 \\ 2 & 2 \end{pmatrix} e^{2t} \frac{\sin(2t)}{2} \\ &= e^{2t} \begin{pmatrix} \cos(2t) - \sin(2t) & -2 \sin(2t) \\ \sin(2t) & \cos(2t) + \sin(2t) \end{pmatrix}. \end{aligned}$$

A general solution is therefore

$$\begin{aligned} x(t) &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \exp(At) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} c_1 [\cos(2t) - \sin(2t)] - 2c_2 \sin(2t) \\ c_1 \sin(2t) + c_2 [\cos(2t) + \sin(2t)] \end{pmatrix}. \end{aligned}$$

- (b)

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution: The characteristic polynomial of the coefficient matrix A is

$$p(z) = z^2 - 3z - 4 = (z + 1)(z - 4).$$

This has the roots $z = -1$ and $z = 4$.

Now there are several approaches you can take. The approach used in class goes as follows. First, compute the exponential matrix $\exp(At)$.

Because A has the two simple real roots -1 and 4 one has that

$$\begin{aligned}\exp(At) &= \frac{1}{4 - (-1)} [(4I - A)e^{-t} + (A + I)e^{4t}] \\ &= \frac{1}{5} \left[\begin{pmatrix} -1 & -3 \\ 2 & 6 \end{pmatrix} e^{-t} + \begin{pmatrix} 6 & 3 \\ -2 & -1 \end{pmatrix} e^{4t} \right] \\ &= \frac{1}{5} \begin{pmatrix} 6e^{4t} - e^{-t} & 3e^{4t} - 3e^{-t} \\ 2e^{-t} - 2e^{4t} & 6e^{-t} - e^{4t} \end{pmatrix}.\end{aligned}$$

A general solution is therefore given by

$$\begin{aligned}x(t) &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \exp(At) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 6e^{4t} - e^{-t} & 3e^{4t} - 3e^{-t} \\ 2e^{-t} - 2e^{4t} & 6e^{-t} - e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} c_1[6e^{4t} - e^{-t}] + c_2[3e^{4t} - 3e^{-t}] \\ c_1[2e^{-t} - 2e^{4t}] + c_2[6e^{-t} - e^{4t}] \end{pmatrix}.\end{aligned}$$

- (6) Sketch phase-plane portraits for each of the following systems. State the type and stability of the origin.

(a)

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Solution: The characteristic polynomial of the coefficient matrix is

$$p(z) = z^2 + z + (-2 - 4) = z^2 + z - 6 = (z + 3)(z - 2).$$

This has the roots $z = -3$ and $z = 2$ with corresponding eigenvectors given by

$$\begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The origin thereby is a *saddle* and is unstable. The portrait is attracting along the line $y = -4x$ and repelling along the line $y = x$.

(b)

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Solution: The characteristic polynomial of the coefficient matrix is

$$p(z) = z^2 + 2z + (1 - 4) = (z + 1)^2 + 4.$$

This has the roots $z = -1 \pm i2$. The origin thereby is a *spiral sink* and is asymptotically stable. The orbits go clockwise around the origin.

(c)

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Solution: The characteristic polynomial of the coefficient matrix is

$$p(z) = z^2 + (-1 + 10) = z^2 + 9.$$

This has the roots $z = \pm i3$. The origin thereby is a *center* and is stable. (It is not asymptotically stable!) The orbits go counterclockwise around the origin.

(7) Consider the system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y^2 \\ x + y + xy \end{pmatrix}.$$

(a) Find all of its critical points.

Solution: Critical points satisfy

$$2x + y^2 = 0, \quad x + y + xy = 0.$$

The first equation above shows that $x = -\frac{1}{2}y^2$. When x is then eliminated from the second equation one finds that

$$-\frac{1}{2}y^2 + y - \frac{1}{2}y^3 = 0.$$

This is solved as

$$0 = y^3 + y^2 - 2y = y(y^2 + y - 2) = y(y - 1)(y + 2),$$

which has roots $y = -2$, $y = 0$, and $y = 1$. The system thereby has three critical points:

$$(-2, -2), \quad (0, 0), \quad \left(-\frac{1}{2}, 1\right).$$

(b) At each critical point, compute the coefficient matrix of the linearization.

Solution: Because

$$\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} 2x + y^2 \\ x + y + xy \end{pmatrix},$$

the matrix of partial derivatives is

$$\begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} 2 & 2y \\ 1 + y & 1 + x \end{pmatrix}.$$

Evaluating this matrix at each critical point yields the coefficient matrices

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \quad \text{at } (-2, -2),$$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{at } (0, 0),$$

$$\begin{pmatrix} 2 & 2 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{at } \left(-\frac{1}{2}, 1\right).$$

This is all that is asked of you. However, if you had been asked to classify the type and stability of each critical point then you can easily see that $(-2, -2)$ is a saddle (the eigenvalues are -2 and 3), $(0, 0)$ is a nodal source (the eigenvalues are 1 and 2), and $(-\frac{1}{2}, 1)$ is also a nodal source (the eigenvalues are $\frac{1}{2}$ and 2). All are therefore unstable.

(8) Suppose you know that for some nonlinear system of differential equations

- the equilibrium solutions are $(1,1)$, $(-2,0)$, and $(-2,3)$;
- for $(1,1)$ the linearization has eigenvalues 3 and 2 with respective eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

- for $(-2,0)$ the linearization has eigenvalues -1 and -3 with respective eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

- for $(-2,3)$ the linearization has eigenvalues -2 and 1 with respective eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

Sketch a plausible phase portrait for the system. Identify the type and stability of each equilibrium solution.

Solution: The type and stability of each equilibrium solution is determined as follows.

- The equilibrium solution $(1,1)$ has two positive simple real eigenvalues. It is thereby a *nodal source* and is unstable.
- The equilibrium solution $(-2,0)$ has two negative simple real eigenvalues. It is thereby a *nodal sink* and is asymptotically stable.
- The equilibrium solution $(-2,3)$ has one negative and one positive simple real eigenvalue. It is thereby a *saddle* and is unstable.

The phase portrait will be sketched during the review.