Lecture 9 – Landau Damping

09/27/17

Happy Birthday, Brittany!

For Homework 3, Problem 3

In the fluid equations

$\gamma_s = 1 \Rightarrow$ the isothermal limit

$$\frac{dn}{dt} + n\nabla\cdot\vec{u} = 0$$

$$\frac{dP}{dt} + P\gamma_s\nabla\cdot\vec{u} = 0$$

$\gamma_s = 1$ makes the pressure equation identical to the continuity equation

$$\frac{dP}{dt} + P\nabla\cdot\vec{u} = 0$$

$\Rightarrow \tilde{p} \sim \tilde{n}T_0$

$\tilde{n}$: density perturbation

$\tilde{p}$: pressure perturbation

Kinetic Dispersion Relation

Starting with the kinetic theory form of the dielectric function (finite temperature, T; warm plasma)

$$\varepsilon(\omega,\vec{k}) = 1 + \frac{\omega_{pe}^2}{n_0 k^2}\int d\vec{v}\,\frac{1}{\omega - \vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0$$

— lacks time dependence!

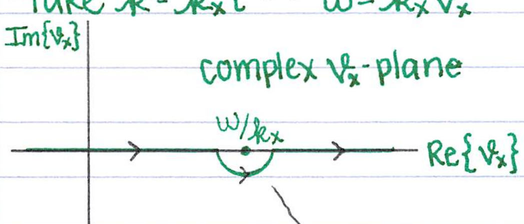
* What about the singularity where $\omega = \vec{k}\cdot\vec{v}$?

To address:

Where does the velocity-space integral go with respect to the singularity?

↳ we must think about the velocity-space integral as an integral in the complex plane

- take $\vec{k} = k_x\hat{i} \rightarrow \omega = k_x v_x$



go under the pole with contour ⇒ but why?

→ Integration under the pole tells us that ε is a complex function; gives rise to damping!

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Landau Damping

Our previous derivation of the dispersion relation lost all causality
↳ must now use a more careful approach ↴

⇒ Solve the Vlasov/Poisson system (& linearize)
- take the Fourier Transform in space
  - doesn't do much
- take the Laplace Transform in time
  - yields causality
  - determines the contour/what direction to go on the contour

Start with:
- the linearized Vlasov Equation

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\nabla f_1 + \frac{e}{m}\nabla\varphi_1\cdot\frac{\partial}{\partial\vec{v}} f_0 = 0$$ *for e^-

& Poisson's Equation

$$\nabla^2\varphi_1 = 4\pi e n_1$$

Fourier Transform – space

$\int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}(\ \ ) = 0$

$$\frac{\partial}{\partial t} f_{1k} + i\vec{k}\cdot\vec{v} f_{1k} + \frac{e}{m}\varphi_{1k}\, i\vec{k}\cdot\frac{\partial}{\partial\vec{v}} f_0 = 0 \qquad \text{eq. } 1$$

$$-k^2\varphi_{1k} = 4\pi e n_{1k}$$

Laplace Transform – introduces direction in time into the problem

*Under Laplace Transformation, differential equations become algebraic equations

General form:

$$g_s = \int_0^\infty dt\, g(t) e^{-st}$$

└ where the infinite integral $\int_0^\infty g(t)dt$ need not exist; it may diverge for exponentially large t
(g_s MUST converge for our purposes)

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Here, we will use an alternate form...

$$g(\omega) = \int_0^\infty dt\, g(t)\, \underbrace{e^{i\omega t}}$$

$$\mathrm{Im}\{\omega\} = \eta \rightarrow e^{i\omega t} \sim e^{-\eta t}$$

To guarantee convergence, $\mathrm{Im}\{\omega\}$ must be sufficiently positive

– What do we mean by "sufficiently positive"?

- {from Arfken & Weber} If there are some constants s_0, M, and t_0, all ≥ 0, such that for all $t > t_0$

$$|e^{-s_0 t} g(t)| \leq M$$

the Laplace transform will exist for $s > s_0$

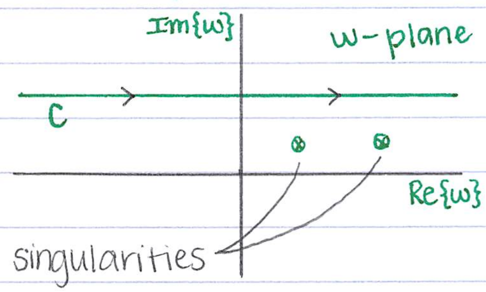
$$\Rightarrow \underbrace{g(t)}_{e^{\gamma t}} e^{i\omega t} \longrightarrow 0 \text{ as } t \rightarrow \infty$$

If g has exponentially-growing roots γ, we require $\mathrm{Im}\{\omega\} > \gamma$ for the LT to converge

*Note: By guaranteeing convergence, that is, demanding that the functions be bounded, we are able to perform the LT even for plasmas with instabilities

NOW: Inverse transform

$$g(t) = \int_C \frac{d\omega}{2\pi}\, g(\omega)\, e^{-i\omega t}$$



$\Rightarrow$ The contour C must be above all singularities of $g(\omega)$ for $t < 0$ and below all singularities of $g(\omega)$ for $t > 0$

for $t < 0$ ($\mathrm{Im}\{\omega\} > 0$ & $g(t) = 0$)

↳ must close contour from above (or to the right, in the s-plane)

$\underbrace{e^{+i\omega(-t)}}$

We need this term to go to zero to satisfy our convergence requirement above, and it would blow up in the lower-half plane (that is, below any singularities)

– any singularities in the upper-half plane would give a

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finite contribution to g(t), and using a Laplace transform we demand their contributions = 0 for t<0

Take the Laplace Transform of eq. ①

$$\int_0^{\infty} dt\, e^{i\omega t}\frac{\partial}{\partial t} f_{1k}(t) = \underbrace{\overbrace{e^{i\omega t} f_{1k}(t)\Big|_0^{\infty}}^{\text{contribution @ } \infty = 0 \text{ because } \mathrm{Im}\{\omega\} \text{ sufficiently positive to kill it}}}_{\text{additional term}} - \underbrace{\int_0^{\infty} dt\, i\omega\, e^{i\omega t} f_{1k}(t)}_{\text{the transform that we wanted}}$$

$$= -\underbrace{f_{1k}(t=0)}_{\text{initial condition}} - i\omega f_{1k\omega} \quad ; \quad f_{1k\omega} = \text{LT of } f_{1k}$$

⇒ transformed Vlasov & Poisson eqns.

$$-i(\omega - \vec{k}\cdot\vec{v}) f_{1k\omega} = -\frac{e}{m}\varphi_{1k\omega}\, i\vec{k}\cdot\frac{\partial}{\partial\vec{v}} f_0 + f_{1k}(t=0)$$

$$k^2 \varphi_{1k\omega} = -4\pi e \int d\vec{v}\, f_{1k\omega}$$

Combining the above & the kinetic dispersion relation...

$$k^2 \varepsilon(\omega,\vec{k})\varphi_{1k\omega} = -4\pi e i \int d\vec{v}\, \frac{f_{1k\omega}(t=0)}{\omega - \vec{k}\cdot\vec{v}} \qquad \text{eq. ②}$$

$$\varepsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{k^2 n_0}\int d\vec{v}\, \frac{\vec{k}\cdot\partial/\partial\vec{v}\, f_0}{\omega - \vec{k}\cdot\vec{v}} = 1 + 4\pi\chi_e$$

linear theory

Now we can perform an inverse transform!

Rearranging eq. ②

$$\varphi_{1k\omega} = -4\pi e i \int d\vec{v}\, \frac{f_{1k\omega}(t=0)}{(\omega - \vec{k}\cdot\vec{v})k^2\varepsilon(\omega,\vec{k})}$$

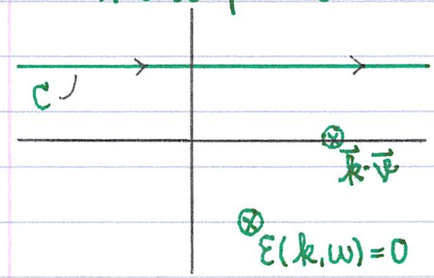
IFT — integral over contour C in the ω-plane

$$\varphi_{1k}(t) = \int_C \frac{d\omega}{2\pi}\varphi_{1k\omega} e^{-i\omega t}$$

$$\varphi_{1k}(t) = -\int_C \frac{d\omega}{2\pi} 4\pi e i \int d\vec{v}\, \frac{f_{1k}(t=0)\, e^{-i\omega t}}{\underbrace{(\omega - \vec{k}\cdot\vec{v})}_{\text{pole for } \omega = \vec{k}\cdot\vec{v}} k^2 \underbrace{\varepsilon(k,\omega)}_{\text{pole for } \varepsilon(k,\omega)=0}} \qquad \text{eq. ③}$$

→

In the ω-plane



→ Contour must lie above all singularities of $\varphi_{1k}(t)$

- For $t<0$: we close the contour in the upper half plane, but it contains no singularities so we find $\varphi_{1k}(t)=0$

- For $t>0$: we close the contour in the lower half plane and pick up contribution from two singularities
  $\omega=\vec{k}\cdot\vec{v}$ and $\varepsilon(k,\omega_0)=0$
  ⇒ Yields $-2\pi i$ times residue

For $t>0$ in the LHP, eq. ③ becomes

$$\varphi_{1k}(t)=\frac{-4\pi e}{k^2}\left[\overbrace{\frac{e^{-i\omega_0 t}}{\partial\varepsilon/\partial\omega_0}\int d\vec{v}\,\frac{f_{1k}(t=0)}{\omega_0-\vec{k}\cdot\vec{v}}}^{\text{natural mode}}+\underbrace{\int d\vec{v}\,\frac{e^{-i\vec{k}\cdot\vec{v}t}}{\varepsilon(k,\vec{k}\cdot\vec{v})}f_{1k}(t=0)}_{\text{addition from free-streaming}}\right]$$

— The first term is the natural mode of the system which evolves from the initial perturbation

— The second term arises from the free-streaming of the particles. It is not a natural mode of the system since $\varepsilon(k,\vec{k}\cdot\vec{v})\neq 0$
  ↳ For large t the rapid oscillations of $\exp\{-i\vec{k}\cdot\vec{v}t\}$ as the velocity integral is carried out cause this term to be small such that it may be neglected

Let's focus on the normal mode of this system ($t>0$)
↳ returning to $\varepsilon(k,\omega)$ and inspecting its singularity...

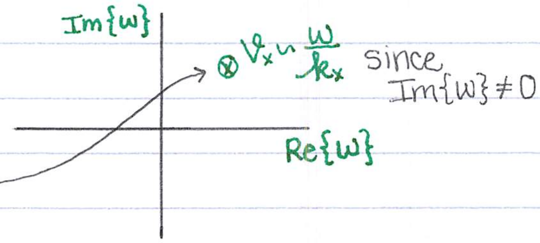
Recall — $\varepsilon(k,\omega)$ is defined such that ω is above all the zeroes of $\varepsilon_{k\omega}$

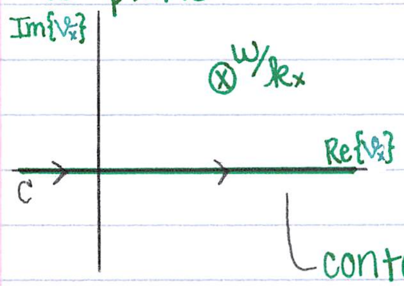
- for simplicity, take $\vec{k}=k_x\hat{\imath}$

$$\Rightarrow \varepsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{k_x^2 n_0} \int d\vec{v}\, \underbrace{\frac{k_x \frac{\partial}{\partial v_x} f_0}{\omega - k_x v_x}}$$

In velocity-space, we have a singularity when $\omega = \vec{k}\cdot\vec{v} \equiv k v_{\parallel} \rightarrow v_x = \omega/k_x$ $(k_x > 0)$

BUT! ω has a complex positive component, putting the singularity in Quadrant 1 of the ω-plane





For solution of the dispersion relation $\varepsilon(k,\omega_0) = 0$

$\hookrightarrow \underbrace{\mathrm{Im}\{\omega_0\} > 0}$ aka the growing mode

contour away from from the singularity → okay (because of the behavior of the function in the upper half plane)

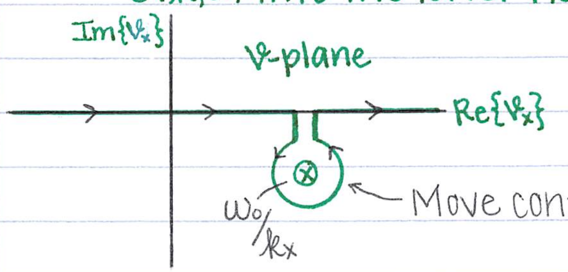
What about the $\mathrm{Im}\{\omega_0\} < 0$ case?

- Not unrealistic! Typical of a damped system

This would move the singularity below the Re{v_x}-axis

*BUT it cannot cross the contour!

$\Rightarrow$ Must deform contour to analytically continue $\varepsilon(k,\omega)$ into the lower half plane



Move contour down with the singularity

Math Aside: Residue Method – Moving a contour up/down with the singularity

* In complex analysis, the residue is a complex number

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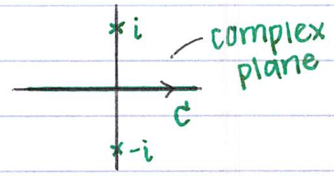
proportional to the contour integral of a meromorphic function along a path enclosing one of its singularities.

Ex:

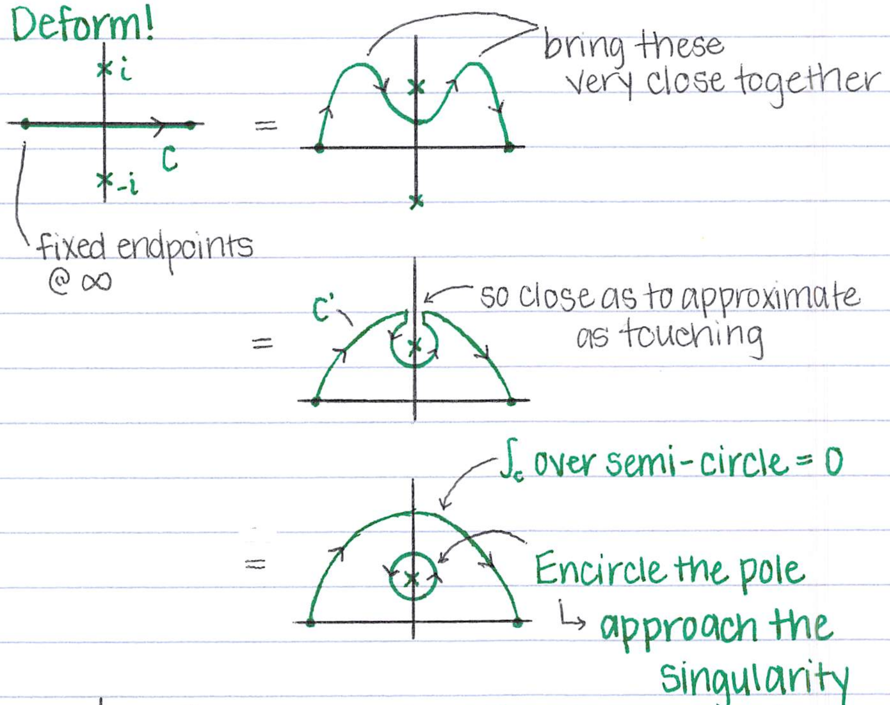
$$I = \int_C \frac{dz}{\underbrace{1+z^2}_{1+z^2=(z+i)(z-i)}}$$

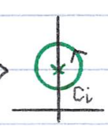
poles



→ **We need contribution from one pole to get a non-zero value from our contour integral**
(deforming the contour to circle both poles would cancel out their contribution)

Deform!

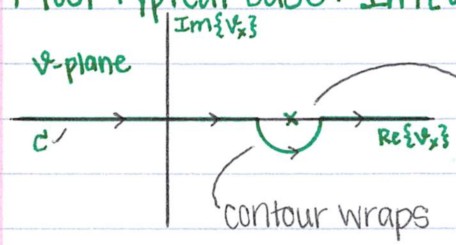


⇒  $\int_{C'} \frac{dz}{(z+i)(z-i)} \xrightarrow[z\to i]{\lim} \int_{C_i} \frac{dz}{2i(z-i)} = \frac{1}{2i} \underbrace{\oint_{C_i} \frac{dz}{(z-i)}}_{=2\pi i} = \pi$

* Make sure to pay attention to contour direction
- integration over CCW contour (like here) > 0, CW direction < 0

** Prescription: start by assuming that $Im\{w_0\} > 0$. If not, deform the contour downwards **

Most typical case: $\mathrm{Im}\{\omega_0\} \simeq 0$



Wrap contour (in v-space) around the singularity — this leads to damping of waves (i.e., Landau damp.)

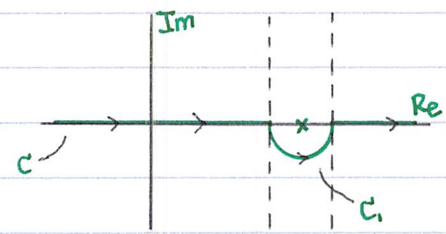
contour wraps down because of Laplace Transform requirements ($t>0$)

$\Rightarrow$ This becomes the Laplace Transform problem

How to handle $\mathrm{Im}\{\omega_0\} \simeq 0$

We want to separate the semi-circle right around the singularity from the rest of the contour

$\rightarrow$ call it C_1



• First, integrate over the deformed C_1

$$\int_{C_1} d\vec{v}\, \frac{\vec{k}\cdot \partial/\partial\vec{v}\, f_0}{\omega - \vec{k}\cdot\vec{v}} = \int_{C_1} dv_x \iint dv_y dv_z \frac{\cancel{k_x}\, \partial/\partial v_x f_0}{-\cancel{k_x}(v_x - \omega/k_x)}$$

take $\vec{k} = k_x\hat{\imath} \rightarrow v_x = v_\parallel;\ v_y, v_z = v_\perp$

$$= -\iint dv_y dv_z \int_{C_1} dv_x \frac{\partial f_0/\partial v_x}{v_x - \omega/k_x}$$

analytic in vicinity of pole

* taking $\partial f_0/\partial v_x$ to be analytic in the vicinity of the singularity allows us to Taylor expand

expansion of $\partial f_0/\partial v_x$ about $v_x = \omega/k_x$:

$$\frac{\partial f_0}{\partial v_x} \approx \left.\frac{\partial f_0}{\partial v_x}\right|_{\omega/k_x} + \underbrace{\left.\frac{\partial^2 f_0}{\partial v_x^2}\right|_{v_x=\omega/k_x}\left(v_x - \frac{\omega}{k_x}\right)}_{\rightarrow 0} + \dots$$

no contribution from higher order terms

(no contribution because $\omega/k_x \sim v_x;\ v_x - v_x = 0$)

$\Rightarrow$ Because $\partial f_0/\partial v_x$ is analytic we can replace it with its value at the singularity

$\downarrow$ plugging this all back in $\downarrow$

$$= -\iint dv_y dv_z \left.\left(\frac{\partial f_0}{\partial v_x}\right)\right|_{\omega/k_x} \int_{C_1} dv_x \frac{1}{v_x - \omega/k_x}$$

The remaining contour integral over C_1 is the integral over the residue

$$\int_{C_1} dv_x \frac{1}{v_x - \omega/k_x} = i\pi$$

residue from the singularity

Recalling the result from combining our transformed Vlasov and Poisson equations...

$$\varepsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{k^2 n_0} \underbrace{\int d\vec{v} \frac{\vec{k}\cdot \partial/\partial\vec{v} f_0}{\omega - \vec{k}\cdot\vec{v}}}_{\text{want to plug in for this integral}}$$

Plugging in our result above yields the <u>Reduced Distribution F'n</u>

$$\Rightarrow \varepsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{k^2 n_0}\left[P\int d\vec{v} \frac{\vec{k}\cdot \partial/\partial\vec{v} f_0}{\omega - \vec{k}\cdot\vec{v}} \underbrace{- i\pi \overbrace{\frac{k_x}{|k_x|}}^{\text{changes sign with } k_x} \iint dv_y dv_z \frac{\partial f_0}{\partial v_x}}_{\text{contribution from resonant particles to the dispersion of } f_0} \right]$$

*P = principal value integral — the integral up to & just after the singularity residue

Note: We would follow the same prescription for $\mathrm{Im}\{\omega\} > 0$

NOW: Calculate damping rate for a plasma wave

Evaluate

$$P\int d\vec{v} \frac{\vec{k}\cdot \partial/\partial\vec{v} f_0}{\omega - \vec{k}\cdot\vec{v}}$$

↓ integrate by parts ↓

$$= \frac{-k^2 n_0}{\omega^2}$$, ignoring resonant particles

answer valid for $\omega/k_x \gg v_{th}$

(This is the cold plasma assumption, valid as long as v_{th} is smaller than the wave phase velocity, $v_p = \omega/k$)

plugging into the reduced distribution function...

$$\varepsilon = 1 + \frac{\omega_{pe}^2}{n_o k^2}\left[\frac{-k^2 n_o}{\omega^2} - i\pi \frac{k_x}{|k_x|} \frac{\partial(\int dv_y dv_z f_o)}{\partial v_x}\bigg|_{\omega/k_x}\right]$$

$$= \underbrace{\varepsilon_R}_{\text{real}} + i\overbrace{\varepsilon_I}^{\text{imaginary}}$$

$$= \underbrace{1 - \frac{\omega_{pe}^2}{\omega^2}}_{\substack{\text{usual} \\ \text{plasma} \\ \text{wave}}} - \underbrace{i\pi \frac{k_x}{|k_x|} \frac{1}{k^2} \frac{\omega_{pe}^2}{n_o} \frac{\partial f_o}{\partial v_x}\bigg|_{\omega/k_x}} = 0 \qquad \text{eq. 4}$$

$\Rightarrow$ We want to know how big this damping contribution term is compared to everything else (under the v_{th} vs. v_p limit)

Say $f_o \sim \frac{1}{v_t} e^{-v_p^2/v_t^2}$

normalization of distribution

damping term $\sim \underbrace{\frac{\omega_{pe}^2}{k^2}}_{\text{like } v_p^2} \frac{v_p}{v_t^2} \frac{e^{-v_p^2/v_t^2}}{v_t}$

$\sim \underbrace{\frac{v_p^3}{v_t^3} e^{-v_p^2/v_t^2}}$, where $\frac{v_p}{v_t} \gg 1$

$\Rightarrow \ll 1$, small value = weak damping

Taking the weak damping into account allows us to...

- write $\omega = \omega_o + \underbrace{\delta\omega}_{\text{shift in } \omega \text{ from damping}}$
- expand in ε

We want to determine $\omega \rightarrow$ solve $\varepsilon(\omega, \vec{k})$

Write

$$\omega = \overbrace{\omega_o}^{\text{lowest order}} + \delta\omega$$

$$\varepsilon = \varepsilon_o + \underbrace{\varepsilon_1}_{\text{first order}} = 0$$

i.e., order by order, each thing must be zero

$$\varepsilon_o = 1 - \frac{\omega_{pe}^2}{\omega_o^2} = 0 \Rightarrow \omega_o = \omega_{pe}$$

just the dielectric constant for high frequency & small ion response (Lec #2) must consist of two pieces (not just ω_0 now, also $\delta\omega$)

$$\varepsilon_1 = \frac{\partial \varepsilon_0}{\partial \omega_0}\delta\omega + i\varepsilon_I$$

$$\delta\omega = \frac{-i\varepsilon_I}{\partial\varepsilon_0/\partial\omega_0} \longrightarrow \text{for } \varepsilon_0 = 1 - \omega_{pe}^2/\omega_0^2$$

$$\left.\frac{\partial\varepsilon_0}{\partial\omega}\right|_{\omega_0} = \left.\frac{2\omega_{pe}^2}{\omega^3}\right|_{\omega_0} = \frac{2}{\omega_{pe}} \quad (\omega_0 = \omega_{pe})$$

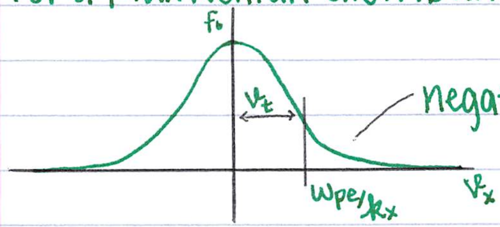
↓ plugging all this into eq. ④ and solving for $\delta\omega$ ↓

$$\Rightarrow \delta\omega = i\frac{\omega_{pe}}{2}\pi\frac{k_x}{|k_x|}\frac{1}{k_x^2}\frac{\omega_{pe}^2}{n_0}\underbrace{\left.\frac{\partial f_0}{\partial v_x}\right|_{\omega_{pe}/k_x}}_{\text{slope (negative)}} = i\frac{\pi}{2}\omega_{pe}\frac{k}{|k_x|}\frac{v_p^2}{n_0}\left.\frac{\partial f_0}{\partial v_x}\right|_{v_p}$$

* The expression for Landau damping of a plasma wave *

Note: this could also represent growth if the distribution function has the opposite sign

For a Maxwellian distribution

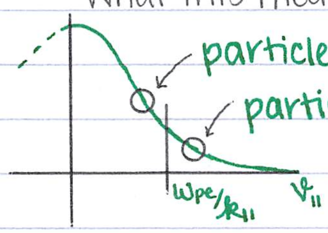


negative slope @ $v_x = \omega_{pe}/k_x$ → damped wave

– But why does this cause the wave to be damped?

Damping occurs because particles with $v_\parallel \simeq v_p$ (here, $v_x \simeq v_p$) see a nearly DC electric field. Particles slightly slower than the wave will gain energy from the wave while those slightly faster will slow down and give energy to the wave. This yields a greater number of particles going slower than v_p, so the wave is damped.

└ what this means visually... ↴



particles here slow down and give energy to the wave

particles here gain energy from the wave and speed up at the expense of the wave

→ More particles take energy than give it, so the wave is damped

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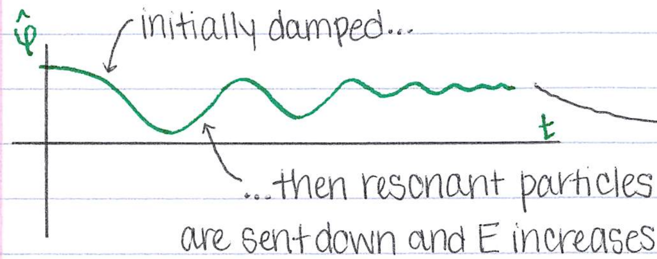
This is reversible, like all processes to do with the Vlasov equation

↳ damping somewhat fictitious; will eventually oscillate



for $\rightarrow v_x \rightarrow v_p$

We eventually get a net energy loss from resonant particles "stuck in the trap"

Plasma Dispersion Function

It is useful to define a standard function that can be used to represent the kinetic plasma dispersion relation for a Maxwellian distribution.

In velocity space:

$$\varepsilon(k,\omega) = 1 + \frac{4\pi q^2}{mk^2}\int d\vec{v}\,\frac{\vec{k}\cdot\partial/\partial\vec{v}\,f_0}{\omega - \vec{k}\cdot\vec{v}}$$

*general form

└ let $\vec{k} = k\hat{z}$

$$f_0 = \text{Maxwellian} = \frac{n_0}{(2\pi T/m)^{3/2}}\, e^{-\frac{1}{2}m\underbrace{(v_x^2+v_y^2+v_z^2)}_{v^2 = v_\perp^2 + v_z^2}/T}$$

· Perform the $v_\perp$ integration

$$\varepsilon(k,\omega) = 1 + \frac{4\pi q^2}{mk^2}\int dv_z \underbrace{\iint dv_x dv_y}_{\int dv_\perp}\,\frac{k\cdot\partial/\partial v_z\, f_0}{\omega - kv_z} \longrightarrow \vec{k}\cdot\partial/\partial\vec{v}\, f_0 = -\tfrac{1}{T}\,\vec{k}\cdot\vec{v}\, f_0 = -\tfrac{1}{T}\, kv_z f_0$$

$\frac{1}{2}m/T = 1/v_t^2$

$$= 1 - \frac{4\pi q^2}{mk^2}\,\frac{n_0}{(2\pi T/m)^{1/2}}\int dv_z\,\frac{kv_z}{T}\,m\,\frac{e^{-\frac{1}{2}mv_z^2/T}}{\omega - kv_z}$$

then the integral over v_z is just

$+\omega-\omega$ to match denominator

$$\varepsilon(k,\omega) = 1 - \frac{4\pi q^2 n_0}{k^2\underbrace{(2\pi T/m)^{1/2}}_{=\sqrt{\pi}\,v_t}}\int dv_z\,\frac{kv_z}{T}\,\frac{e^{-v_z^2/v_t^2}}{\omega - kv_z}$$

→

$$\varepsilon(k,\omega) = 1 - \underbrace{\frac{4\pi q^2 n_0}{k^2 T}}_{k_D^2 = \frac{4\pi n q^2}{T}} \int dv_z \, \frac{1}{\sqrt{\pi}\, v_t} \, \underbrace{\frac{kv_z - \omega + \omega}{\omega - kv_z}}_{= -(kv_z - \omega)} \, e^{-v_z^2/v_t^2}$$

$$= 1 + \frac{k_D^2}{k^2}\left[1 + \int \frac{dv_z}{\sqrt{\pi}\, v_t} \frac{\omega}{kv_z - \omega} e^{-v_z^2/v_t^2}\right]$$

BUT! This is not a generalized standard function

→ Define:

$$s \equiv \frac{v_z}{v_t}, \quad \zeta \equiv \frac{\omega}{k_z v_t}$$

a tabulated function in the complex plane

Then using this terminology...

$$\varepsilon = 1 + \frac{k_D^2}{k^2}\Big(1 + \zeta \underbrace{\int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{s - \zeta}}_{\text{The Plasma Dispersion Function}}\Big)$$

<u>The Plasma Dispersion Function</u>

a.k.a. the Z-function

$$Z(\zeta) = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{s - \zeta}, \quad \text{defined for } \operatorname{Im}\{\zeta\} > 0$$

Such that

$$\varepsilon = 1 + \frac{k_D^2}{k^2}\left(1 + \zeta Z(\zeta)\right)$$