Lecture 8 - Kinetic & Fluid Theories

09/21/17

Kinetic Theory of Electrostatic Waves

We want to solve the Vlasov equation to explore what happens in a warm plasma.

→ Start with the linearized Vlasov equation:

$$(-i\omega + i\vec{k}\cdot\vec{v})\hat{f} - \frac{q}{m}\hat{\varphi}\, i\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0 = 0$$

want to solve this for $\hat{f}$ in terms of $\hat{\varphi}, f_0$

Where

$$f = f_0 + f_1 + \dots$$

$$f_1 = \mathrm{Re}\{\hat{f}e^{i\vec{k}\cdot\vec{x} - i\omega t}\}$$

$$\vec{E}_1 = -\nabla\varphi_1$$

$$\varphi_1 = \mathrm{Re}\{\hat{\varphi}_1 e^{i\vec{k}\cdot\vec{x} - i\omega t}\}$$

↓ rearranging linear Vlasov ↓

$$(-i\omega + i\vec{k}\cdot\vec{v})\hat{f} = \frac{q}{m}\hat{\varphi}\, i\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0$$

$$\hat{f} = \frac{q/m\,\hat{\varphi}\,\cancel{i}\vec{k}\cdot\partial/\partial\vec{v}\,f_0}{-\cancel{i}\omega + \cancel{i}\vec{k}\cdot\vec{v}}$$

$$\Rightarrow \hat{f} = \frac{q/m\,\hat{\varphi}\,\vec{k}\cdot\partial/\partial\vec{v}\,f_0}{\underbrace{\vec{k}\cdot\vec{v} - \omega}_{\text{Doppler shift}}}$$

<u>Perturbed distribution function</u> in the presence of the $\vec{E}$-field

$$= -\frac{q}{m}\hat{\varphi}\frac{1}{\underbrace{\omega - \vec{k}\cdot\vec{v}}}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0$$

$\omega - \vec{k}\cdot\vec{v}$ = effective frequency seen by moving particles

NOW: Need to calculate the charge perturbation and put that into Poisson's equation to calculate φ_1

charge density

$$\rho_1 = \mathrm{Re}\{\hat{\rho}e^{i\vec{k}\cdot\vec{x} - i\omega t}\}$$

$$\hat{\rho} = \sum_{i,e} q\int d\vec{v}\,\hat{f}_{e,i}$$

$$= -\sum_{i,e}\int d\vec{v}\,\underbrace{\frac{q^2}{m}\,\frac{\hat{\varphi}\,\vec{k}\cdot\partial/\partial\vec{v}\,f_0}{\omega - \vec{k}\cdot\vec{v}}}_{=q\hat{f}}$$

KLE

Manipulating Poisson's equation...

$\nabla \cdot \vec{E}_1 = 4\pi \rho_1$

$i\vec{k} \cdot \hat{\vec{E}} = 4\pi \hat{\rho}$

└ $\hat{\vec{E}} = -i\vec{k}\hat{\varphi}$

$k^2 \hat{\varphi} - 4\pi \hat{\rho} = 0$

↳ plugging in yields ↴

$$\underbrace{k^2 \hat{\varphi}} + 4\pi \sum_{i,e} \int d\vec{v}\, \frac{q^2}{m} \frac{\hat{\varphi}\, \vec{k} \cdot \partial/\partial \vec{v}\, f_0}{\omega - \vec{k} \cdot \vec{v}} = 0$$

└ factor out $k^2 \hat{\varphi}$

$$k^2 \underbrace{\left(1 + \sum_{e,i} \frac{4\pi q^2}{m k^2} \int d\vec{v}\, \frac{1}{\omega - \vec{k} \cdot \vec{v}}\, \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0 \right)}_{\varepsilon(\vec{k},\omega)} \hat{\varphi} = 0$$

$$\Rightarrow \varepsilon(\vec{k},\omega) = 1 + \sum_{e,i} \frac{1}{k^2} \frac{4\pi q^2}{m} \int d\vec{v}\, \frac{1}{\omega - \vec{k} \cdot \vec{v}}\, \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0$$

The dielectric function for a warm plasma (finite T)

$\rightarrow \varepsilon(\vec{k},\omega) = 0$ for waves

*Check to see that we obtain the cold plasma result under the appropriate limits:

↓ integrate by parts ↓

$$\varepsilon = 1 + \sum_{e,i} \frac{1}{k^2} \frac{4\pi q^2}{m} \int d\vec{v}\, \underbrace{\frac{1}{\omega - \vec{k} \cdot \vec{v}}}\, \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0$$

$$\vec{k} \cdot \frac{\partial}{\partial \vec{v}} \frac{1}{\omega - \vec{k} \cdot \vec{v}} = + \frac{1}{(\omega - \vec{k} \cdot \vec{v})^2}\, \underbrace{\vec{k} \cdot \frac{\partial}{\partial \vec{v}} (+\vec{k} \cdot \vec{v})}$$

$$\vec{k} \cdot \frac{\partial}{\partial \vec{v}} (\vec{k} \cdot \vec{v}) = k_i \frac{\partial}{\partial v_i} k_i v_i$$

$$= k_i k_j \delta_{ij}$$

$$= k_i^2 = k^2$$

$$= 1 + \sum_{e,i} \frac{1}{\cancel{k^2}} \frac{4\pi q^2}{m} \int d\vec{v}\, f_0 \frac{\cancel{k^2}}{(\omega - \vec{k} \cdot \vec{v})^2}$$

$$= 1 + \sum_{e,i} \frac{4\pi q^2}{m} \int d\vec{v}\, \frac{f_0}{(\omega - \vec{k} \cdot \vec{v})^2}$$

Now, in the cold plasma limit

$k v_t \ll \omega \rightarrow \underbrace{v_t \ll \omega/k = v_p}$ — phase velocity

Thermal velocity small compared with the phase speed of the wave $\Rightarrow$ drop $\vec{k}\cdot\vec{v}$

$$\varepsilon = 1 - \sum_{e,i} \frac{4\pi q^2}{m} \underbrace{\int d\vec{v} f_0}_{n_0} \frac{1}{\omega^2}$$

$$= 1 - \sum_{e,i} \frac{4\pi n_0 q^2}{m} \frac{1}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}$$

This is what we got before, from fluid theory! (Lec. #2)

→ Approximation is valid when $\omega \gg k v_{th}$, so if $v_p \gg v_{th}$, the cold plasma limit is okay

phase speed — thermal speed

Thermal Corrections

Want to include thermal corrections to this dispersion relation

Take this to be small; Taylor expand

$$\frac{1}{(\omega - \vec{k}\cdot\vec{v})^2} \sim \frac{1}{\omega^2}\left(1 + \frac{2\vec{k}\cdot\vec{v}}{\omega} + \frac{3(\vec{k}\cdot\vec{v})^2}{\omega^2} + \dots\right)$$

such that

$$\int d\vec{v} \frac{f_0}{(\omega - \vec{k}\cdot\vec{v})^2} \simeq \frac{1}{\omega^2}\left[n_0 + \cancel{\frac{0}{\omega}} + \frac{3}{\omega^2} \underbrace{\int d\vec{v} (\vec{k}\cdot\vec{v})^2 f}\right]$$

$$\frac{1}{2} m \int d\vec{v} (\vec{k}\cdot\vec{v})^2 f = \frac{1}{2} n_0 T k^2$$

forgetting about ions

$$= \frac{1}{\omega^2} n_0 \left(1 + \underbrace{\frac{3}{\omega^2} \frac{T_e k^2}{m}}_{\text{correction term}}\right)$$

↓ plugging back into ε ↓

$$\varepsilon = 1 - \frac{4\pi e^2}{m} \frac{n_0}{\omega^2}\left(1 + \frac{3k^2 T_e}{m_e \omega^2}\right)$$

ω_{pe}^2

$$\varepsilon = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3k^2 \frac{T_e}{m_e \omega^2}\right)$$

$$T_e = \tfrac{1}{2} m_e v_{te}^2$$

$$\varepsilon = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{k^2 v_{te}^2}{\omega^2}\right) \Rightarrow$$ Corrected Dispersion Relation
↳ must solve for ω

– cold plasma assumption valid for

$$\underbrace{k v_{te}}_{} \ll \omega$$

don't need to keep corrections as long as this is small

put another way...

$v_{te} \ll \omega/k = v_p$ → our previous expression is valid so long as thermal velocity is smaller than phase velocity

Solve for ω

To lowest order, $\omega = \omega_{pe}$ (neglects ions)

so:

$$\varepsilon = 1 - \frac{\omega_{pe}^2}{\omega^2}\left[1 + \frac{3}{2}\left(\frac{k v_{te}}{\omega_{pe}}\right)^2\right] \rightarrow \text{Set} = 0,\ \text{solve for } \omega$$

$$\omega^2 = \omega_{pe}^2\left[1 + \frac{3}{2}\left(\frac{k v_{te}}{\omega_{pe}}\right)^2\right]$$

b/c Taylor expansion

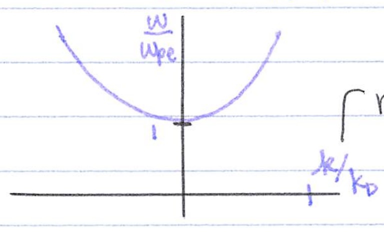
$$\omega = \sqrt{\omega_{pe}^2\left[1 + \frac{3}{2}\left(\frac{k v_{te}}{\omega_{pe}}\right)^2\right]} \approx \omega_{pe}\left[1 + \frac{3}{4}\left(\frac{k v_{te}}{\omega_{pe}}\right)^2\right]$$

BUT:

$$\frac{v_{te}^2}{\omega_{pe}^2} = \frac{\tfrac{1}{2} v_{te}^2 m_e}{4\pi n e^2} = \frac{2T_e}{\underbrace{4\pi n e^2}_{}} = \frac{2}{k_D^2}$$

k_D^2 = the characteristic Debye length scale $= 4\pi n e^2/T$

$$\Rightarrow \omega = \omega_{pe}\left(1 + \frac{3}{2}\frac{k^2}{k_D^2}\right)$$ Plasma wave dispersion: Kinetic theory



require $k/k_D \ll 1$ for expansion to be valid

KLE

Do the fluid equations give the same for a warm plasma?

__Fluid Model__ – only electrons, drop ions

- Continuity Equation ①

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{U}) = 0$$

- Fluid Momentum Equation ②

$$m_e n\left(\frac{\partial}{\partial t}\vec{U} + \vec{U}\cdot\nabla\vec{U}\right) = -\nabla\overline{\overline{P}} - ne\vec{E}$$

$\vec{E} = -\nabla\varphi$

$\nabla\cdot\vec{E} = 4\pi\rho$

- Pressure Equation ③

$$\frac{\partial}{\partial t}\overline{\overline{P}} + \vec{U}\cdot\nabla P + \gamma_s P\nabla\cdot\vec{U} = 0$$

@ Equilibrium: $P_0, n_0, \vec{U}_0 = 0, \vec{E}_0 = 0$

→ We want to linearize!

take $P = P_0 + \mathrm{Re}\{\hat{P}e^{i\vec{k}\cdot\vec{x} - i\omega t}\}$

$\vec{U} = \mathrm{Re}\{\hat{U}e^{i\vec{k}\cdot\vec{x} - i\omega t}\}$

etc.

Linearize to 1st order:

① $-i\omega\hat{n} + n_0 i\vec{k}\cdot\hat{\vec{U}} = 0$

② $m_e n_0(-i\omega\hat{\vec{U}}) = -i\vec{k}\hat{P} - n_0 e\hat{\vec{E}}$

③ $-i\omega\hat{P} + \gamma_s P_0 i\vec{k}\cdot\hat{\vec{U}} = 0$

Note: $\vec{E} = -\nabla\varphi$

$\nabla\cdot\vec{E} = 4\pi\rho$

$\nabla\cdot\vec{E}_1 = -4\pi e n_1$

$k^2\hat{\varphi} = -4\pi e\hat{n}$

Rewriting...

① $n_0 \cancel{i}\vec{k}\cdot\hat{\vec{U}} = \cancel{i}\omega\hat{n}$

$$\vec{k}\cdot\hat{\vec{U}} = \omega\frac{\hat{n}}{n_0}$$

③ $\gamma_s P_0 \cancel{i}\underbrace{\vec{k}\cdot\hat{\vec{U}}}_{\text{plug in eq. ①}} = \cancel{i}\omega\hat{P}$

$$\gamma_s P_0\left(\omega\frac{\hat{n}}{n_0}\right) = \omega\hat{P}$$

$$\hat{P} = \frac{\gamma_s P_0}{\omega}\,\omega\frac{\hat{n}}{n_0}$$

dotting $\vec{k}$ into eq. ②

$$\vec{k}\cdot② \quad \vec{k}\cdot\left(-i\omega n_o m_e \hat{\vec{U}} = -i\vec{k}\hat{P} - n_o e\hat{\vec{E}}\right)$$

$$-i\omega n_o m_e \underbrace{\vec{k}\cdot\hat{\vec{U}}}_{\text{plug in eq. ①}} = \vec{k}\cdot\left(-i\vec{k}\hat{P} - n_o e\hat{\vec{E}}\right)$$

$$-i\omega n_o m_e\left(\omega\frac{\hat{n}}{n_o}\right) = -i\vec{k}\cdot\vec{k}\hat{P} - n_o e\underbrace{\vec{k}\cdot\hat{\vec{E}}}$$

plug in ③
$\hat{P} = \gamma_s P_o \hat{n}/n_o$

$\vec{k}\cdot\hat{\vec{E}} = -ik^2\underbrace{\hat{\varphi}}_{= -4\pi e\hat{n}}$

$$\cancel{-i}\omega m_e(\omega\cancel{\hat{n}}) = \cancel{-i}k^2\gamma_s P_o\frac{\cancel{\hat{n}}}{n_o} + n_o e(\cancel{i})(+4\pi e\cancel{\hat{n}})$$

$$\underbrace{m_e}\omega^2 = k^2\gamma_s P_o\left(\frac{1}{n_o}\right) + 4\pi e^2 n_o$$

divide through $\quad\quad P = nT,\ P_o = n_o T_e$

$$\frac{\cancel{m_e}\omega^2}{\cancel{m_e}} = \frac{k^2\gamma_s\cancel{n_o}T_e}{\cancel{n_o}m_e} + \frac{4\pi e^2 n_o}{m_e}$$

$$\omega^2 = \frac{k^2\gamma_s T_e}{m_e} + \underbrace{\frac{4\pi e^2 n_o}{m_e}}_{\omega_{pe}^2} \Rightarrow \varepsilon \simeq 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{k^2 T_e\gamma_s}{m_e\omega^2}$$

SO:

$$\omega^2 = \omega_{pe}^2\left(1 + \frac{k^2\gamma_s T_e}{m_e\omega_{pe}^2}\right)$$

$$\omega = \sqrt{\omega_{pe}^2\left(1 + \frac{k^2\gamma_s T_e}{m_e\omega_{pe}^2}\right)} = \omega_{pe}\left(1 + \frac{1}{2}\frac{k^2\gamma_s T_e}{m_e\omega_{pe}^2}\right)$$

plug in $\omega_{pe}^2 = \dfrac{4\pi n_o e^2}{m_e}$

$$= \omega_{pe}\left[1 + \frac{1}{2}\frac{k^2\gamma_s T_e}{\cancel{m_e}}\left(\frac{\cancel{m_e}}{4\pi n_o e^2}\right)\right]$$

$= 1/k_D^2$

$$\Rightarrow \omega_F = \omega_{pe}\left(1 + \frac{1}{2}\frac{k^2}{k_D^2}\gamma_s\right)$$

Plasma wave dispersion: Fluid theory

- Compare to kinetic theory result:

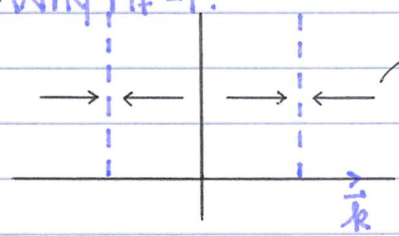
$$\omega_k = \omega_{pe}\left(1 + \frac{3}{2}\frac{k^2}{k_D^2}\right) \rightarrow$$

Fluid theory yields same result if $\gamma_s = 3$

$\gamma_s = \frac{(n_f + 2)}{n_f} = 3$ ← true for $n_f = 1$

⇒ Only okay if you assume the system has 1 degree of freedom

- Why $n_f = 1$?



Oscillatory motion compresses the plasma along $\vec{k}$ → increases internal energy in the $\underbrace{\text{x-direction}}_{\text{not transverse spread}}$

⇒ $\Delta E = \frac{1}{2} m_e \Delta v_x^2$

└ no change in v_y or v_z

- What if $n_f \neq 1$?

In a fluid model with e.g. $n_f = 3$, any increase in $\frac{1}{2} m_e \Delta v_x^2$ spreads to $\frac{1}{2} m_e \Delta v_y^2$ and $\frac{1}{2} m_e \Delta v_z^2$

└ no longer lowest order in kinetic theory here, violating the assumption we made to neglect ions and solve for ω_k.

Landau Damping

Returning to original kinetic theory form

$$\varepsilon = 1 + \frac{\omega_{pe}^2}{n_0 k^2} \int d\vec{v} \underbrace{\frac{1}{\omega - \vec{k} \cdot \vec{v}}}_{\text{Must deal with singularity @ } \omega = \vec{k} \cdot \vec{v}} \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0$$

To address:

Where does the velocity space integral go with respect to the singularity?

↳ We must think about the velocity space integral as an integral in the complex plane

→

KLE

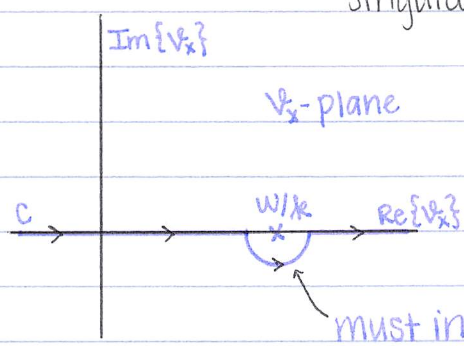
let $\vec{k} = k_x \hat{i}$...

$$\int d\vec{v} \frac{1}{\omega - \vec{k}\cdot\vec{v}} \vec{k}\cdot\frac{\partial}{\partial \vec{v}} f_0 \Rightarrow \int dv_y dv_z \int dv_x \frac{1}{\omega - k_x v_x} k_x \frac{\partial}{\partial v_x} f_0$$

$v_x = \omega / k_x$

This integral is only defined for $Im\{\omega\} > 0$

↳ this tells us that you must stay below the singularity; cannot cross it with our contour



must integrate below the singularity

* This tells us that ε is a complex function!

- ε only defined for $Im\{\omega\} > 0$ -

$\Rightarrow$ Gives rise to damping of plasma waves... "Landau Damping"

To show that this is correct, we will need to go back to the Vlasov equation and find a more careful solution using Laplace Transforms.