# Lecture 6 - The Krook Model & Fluid Equations

09/14/17

from last time...

## Adding Collisions to the Boltzmann Equation

Recall: The Boltzmann equation with collisions

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)f + \frac{\vec{F}}{m}\cdot\frac{\partial}{\partial \vec{v}} f = \left(\frac{\partial f}{\partial t}\right)_c$$

$= C(f)$, the collision operator

$\vec{F}$ = external force; e.g., fields acting on particles

focus on small-angle Coulomb collisions

$\Rightarrow$ Yields the Fokker-Planck equation

$$\left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} = \underbrace{-\frac{\partial}{\partial \vec{v}}\cdot\frac{d}{dt}}_{\text{drag}}\langle\Delta\vec{v}\rangle f + \frac{1}{2}\underbrace{\frac{\partial^2}{\partial\vec{v}\partial\vec{v}}:\frac{d}{dt}}_{\text{diffusion}}\langle\Delta\vec{v}\Delta\vec{v}\rangle f$$

{Drag & Diffusion terms}

$\rightarrow$ process is local in velocity space

(due to small-angle collisions)

- $\frac{d}{dt}\langle\Delta\vec{v}\rangle = \frac{1}{\Delta t}\int d\Delta\vec{v}\, P(\vec{v},\Delta\vec{v})\Delta\vec{v}$
- $\frac{d}{dt}\langle\Delta\vec{v}\Delta\vec{v}\rangle = \frac{1}{\Delta t}\int d\Delta\vec{v}\, \underbrace{P(\vec{v},\Delta\vec{v})}\Delta\vec{v}\Delta\vec{v}$

= probability a particle w/ velocity $\vec{v}$ in velocity space will suffer a jump $\Delta v$ in a time $\Delta t$

we can also use...

## Landau Form of the Collision Model

- looking at the evolution of a species $\alpha$:

$$\frac{\partial}{\partial t}f^\alpha + \vec{v}\cdot\nabla f^\alpha + \frac{\vec{F}}{m_\alpha}\frac{\partial}{\partial\vec{v}}f^\alpha = \sum_\beta \underbrace{C(f^\alpha, f^\beta)}$$

distribution function of species $\alpha$

rate of change of $f^\alpha$ due to collisions with species $\beta$

$C(f^\alpha, f^\beta)$ = the Landau operator (a function of $v$)

*this form introduces nonlinearity in $v$ space

KLE

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→ the Landau operator satisfies all of the relevant properties of $C(f)$

{ HW #2 problem associated with $e^-$–ion collisions
supposed to show Lorentz collision operator }

## The Krook Model

$$C(f) = -\underbrace{\gamma}_{\text{const.}}\left(f - \frac{\overbrace{n(\vec{x})}^{\text{potentially space-dependent density}}}{(2\pi T_0/m)}\, e^{-\frac{1}{2}mv^2/T_0}\right)$$

This form:

- conserves the number density
- can also use a Krook Model that conserves energy and momentum (have to MAKE it do this)

*Major limitation: Does NOT describe small angle scattering, not local in velocity space, and not very realistic — BUT it's simple!

⇒ HW Problem related to linearization w/ a small parameter ⇐
↳ example below

We have the equation

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \underbrace{\frac{q\vec{E}}{m}\cdot\frac{\partial}{\partial \vec{v}} f}_{\text{want to linearize this to lowest order, } f_0} = -\gamma\left(f - \frac{n(\vec{x})}{(2\pi T_0/m)^{3/2}}\, e^{-\frac{1}{2}mv^2/T_0}\right)$$

→ $f_0 = \dfrac{n_0}{(2\pi T_0/m)}\, e^{-\frac{1}{2}mv^2/T_0}$

→ Add small E

How do we linearize?

expand $f$ & $n$:

$$\underbrace{f = f_0}_{0^{th}\text{ order in E}} + \underbrace{\tilde{f}}_{1^{st}\text{ order in E}} + \ldots \quad , \quad n(x) = n_0 + \tilde{n} + \ldots$$

$$\frac{\partial \tilde{f}}{\partial t} + \vec{v}\cdot\nabla\tilde{f} + \frac{q\vec{E}}{m}\cdot\frac{\partial f_0}{\partial \vec{v}} = -\gamma\left(\tilde{f} - \frac{\tilde{n}}{(2\pi T_0/m)^{3/2}}\, e^{-\frac{1}{2}mv^2/T_0}\right)$$

KLE

*HW problem in which we'll have to do this*

## How to solve the collisionless Boltzmann equation

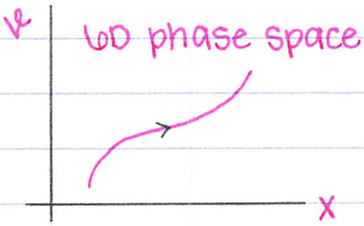
↳ Characteristics of the Vlasov equation

- no collisions / collisionless limit

Characteristic curves are trajectories in phase space of individual particles

$$\frac{\partial f}{\partial t} + \underbrace{\vec{v}}\cdot\nabla f + \underbrace{\frac{\vec{F}}{m}}\cdot\frac{\partial}{\partial \vec{v}} f = 0 \quad \text{OR} \quad \frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt}\cdot\nabla f + \frac{d\vec{v}}{dt}\cdot\frac{\partial}{\partial \vec{v}} f = 0$$

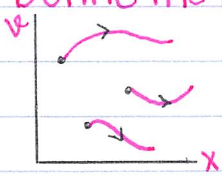
where $\frac{d\vec{x}}{dt} = \vec{v}$, $\frac{d\vec{v}}{dt} = \frac{\vec{F}}{m}$




$\Rightarrow \frac{df}{dt} = 0$, f is a constant along a trajectory (Liouville's thm.; Lec.4)

i.e., patches of phase space density move around such that their values don't change.

- Define the trajectories




Suppose that f has a maximum value $f_{max}$

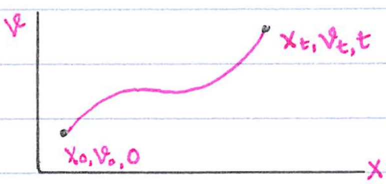
$f_{max} \leq$ for all t

We may define a set of trajectories for all particles:

$x_t = x_t(\vec{x}_0, \vec{v}_0, t)$ → position of particle at time t such that $x = x_0$, $v = v_0$ @ $t = 0$

$v_t = v_t(\vec{x}_0, \vec{v}_0, t)$

$$\begin{cases} \dfrac{d\vec{x}_t}{dt} = \vec{v}_t \\[2ex] \dfrac{d\vec{v}_t}{dt} = \dfrac{F(\vec{x}_t, \vec{v}_t, t)}{m} \end{cases}$$




f is constant so ↴

$f(\vec{x}_t, \vec{v}_t, t) = f(\vec{x}_0, \vec{v}_0, t=0)$

such that we may determine $\vec{x}_t$, $\vec{v}_t$ values for all future time, t, using our constant particle distribution function, f.

Define an inverse problem:

- We know $\vec{x}_t, \vec{v}_t$ at $t$ and we want $\vec{x}_0, \vec{v}_0$ at an earlier time

$$\vec{x}_0 = \vec{x}_0(\vec{x}_t, \vec{v}_t, t)$$
$$\vec{v}_0 = \vec{v}_0(\vec{x}_t, \vec{v}_t, t)$$

We already know that $f(\vec{x}_t, \vec{v}_t, t) = f(\vec{x}_0, \vec{v}_0, t=0)$

Rewrite

$$f(\vec{x}_t, \vec{v}_t, t) = f[\vec{x}_0(\vec{x}_t, \vec{v}_t, t), \vec{v}_0(\vec{x}_t, \vec{v}_t, t), t=0]$$

let $x_t \rightarrow x$

$v_t \rightarrow v$

$$f(\vec{x}, \vec{v}, t) = f[\vec{x}_0(\vec{x}, \vec{v}, t), \vec{v}_0(\vec{x}, \vec{v}, t), 0]$$

This must be a solution to the Vlasov equation

This states that at time $t$, the value of $f$ is given by the phase space location @ $t=0$

In the simple case:

$\vec{F} = 0$ $\rightarrow$ 1-Dimen. $\Big\}$ $x_0 = x - vt$, $v_0 = v$

*this is just one way to find solutions to the Vlasov eqn.

- plugging in

$$f(x, v, t) = f(x - vt, v, 0)$$

such that the value of $f$ @ time $t$ stems directly from $f$ that was @ time $t=0$ at $x - vt$

Another way to find Vlasov solutions...

Solutions in Terms of Constants of Motion

Suppose we have a constant of motion of a particle

ex: energy

$$H = \frac{1}{2}mv^2 + q\varphi(\vec{x})$$

Maxwell-Boltzmann distribution

$$E = -\nabla\varphi, \quad \frac{\partial\varphi}{\partial t} = 0$$

examine classical motion $(\vec{v}(t), \vec{x}(t))$

$$\frac{d}{dt}H = m\vec{v}\cdot\dot{\vec{v}} + q\vec{v}\cdot\nabla\varphi$$

$$\underbrace{\frac{d}{dt}H=0}_{\text{energy is conserved}} \qquad \text{since } m\dot{\vec{v}}=-q\nabla\varphi$$

NOW: Show that $f(\vec{x},\vec{v},t)=f[H(\vec{x},\vec{v})]$ for any $f$ that satisfies the Vlasov equation

$$\frac{\partial f(H)}{\partial t}=0$$ must be since $dH/dt=0$

$$\vec{v}\cdot\nabla f(H)=\vec{v}\cdot\nabla H\frac{\partial f}{\partial H}$$

$$=q\vec{v}\cdot\nabla\varphi\frac{\partial f}{\partial H}$$

$$\underbrace{\frac{q\vec{E}}{m}}_{q\vec{E}=\vec{F}}\cdot\frac{\partial}{\partial\vec{v}}f(H)=\frac{-q}{\cancel{m}}\nabla\varphi\cdot\underbrace{\frac{\partial}{\partial\vec{v}}H}_{=\partial/\partial\vec{v}\left(\frac{1}{2}mv^2+q\varphi(\vec{x})\right)=\cancel{m}\vec{v}}\frac{\partial f}{\partial H}$$

↓ plugging it all in ↓

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\nabla f+\frac{\vec{F}}{m}\cdot\frac{\partial}{\partial\vec{v}}f=0$$

$$\Rightarrow\frac{df}{dt}=0$$ Confirmation that collisions drive $f$ to thermal equilibrium as the distribution as a whole does not change over time

Another example: Canonical Momentum

$$\vec{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B}=\nabla\times\vec{A}$$ ← Vector potential

$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}}{l}d\tau'$$ from the Biot-Savart law

$\vec{A},\varphi$ independent of $x$

↳ therefore ↴

$$p_x=mv_x+q/c\,A_x=\text{const.}$$

$\Rightarrow$ Can write $f$ in terms of $p_x$

$$f=f(p_x,H)$$

*Note: Related to a problem from HW#3 (due in two weeks)

*Description in terms of a fluid can be very useful*
↳ time to derive the fluid equations

## The Fluid Equations {Ref: Bellan 2.4-2.5}

The Vlasov equation with a collision operator can describe the dynamics of a plasma from the collisionless limit all the way to a system with very high collision rates as long as the plasma parameter $\Gamma$ is small (the weakly coupled regime)

- In the limit of large collisions, we expect the dynamics to be fluid-like
- Even in the limit of weak collisions, fluid-like concepts can help us to understand how a plasma behaves

We will proceed to construct a set of fluid equations by taking moments of the Vlasov equation

→ Start with the Vlasov eqn. with a collision operator ←

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla f_\alpha + \frac{\vec{F}}{m_\alpha}\cdot\frac{\partial}{\partial\vec{v}} f_\alpha = \sum_\beta \underbrace{C_{\alpha\beta}(f_\alpha)}_{\text{collisions acting on } \alpha \text{ due to species } \beta}$$

can switch order; $x, v$ independent

*by "moment" → multiply through by $v^n$, integrate over $v$

### Zero Moment (n=0)

- First integrate over the velocity and use the fact that $\partial/\partial\vec{v}\cdot\vec{F} = 0$ and that collisions don't create or destroy particles

Properties of the collisions:

$\int d\vec{v}\, C_{\alpha\beta}(f_\alpha) = 0$

└ Coulomb collisions conserve particle number (from Lec. #5)

and $f_\alpha \to 0$ as $\underbrace{v \to \infty}$

from Lec. #3, particles moving too quickly have a greatly reduced interaction cross-section

$$\underbrace{\int d\vec{v}\, \vec{F}(\vec{x},\vec{v},t)\cdot\frac{\partial}{\partial\vec{v}} f_\alpha}=0$$

$$\frac{\partial}{\partial\vec{v}}\cdot\vec{F}=0 \longrightarrow \frac{\partial}{\partial\vec{v}}\cdot(q\vec{E}+\vec{v}\times\vec{B}) = 0+\frac{\partial}{\partial\vec{v}}\cdot(\vec{v}\times\vec{B})$$

$$=\vec{B}\cdot\underbrace{\left(\frac{\partial}{\partial\vec{v}}\times\vec{v}\right)}_{0}=0$$

This leaves us with

$$\frac{\partial}{\partial t}\int d\vec{v} f_\alpha + \nabla\cdot\underbrace{\int d\vec{v}\,\vec{v} f_\alpha}_{n_\alpha\vec{u}_\alpha} = 0$$

$$\Rightarrow \frac{\partial n_\alpha}{\partial t}+\nabla\cdot n_\alpha\vec{v}_\alpha = 0$$ } Continuity Equation

$n_\alpha \equiv \int d\vec{v} f_\alpha$

$n_\alpha\vec{u}_\alpha \equiv \int d\vec{v}\,\vec{v} f_\alpha$ ⇒ defines $\vec{u}_\alpha$, the mean drift velocity of plasma species $\alpha$

*The Continuity Equation states that the change in number density of particles at a location arises from the flux of particles in/out of the region.

## First Moment (n=1)

Multiply the Vlasov equation by $\vec{v}$ and integrate

→ Note: $\vec{v}$ is an independent variable and so passes through $\nabla$, $\partial/\partial t$ operators

$$\frac{\partial}{\partial t}\int d\vec{v} f_\alpha\vec{v} + \nabla\cdot\int d\vec{v}\,\vec{v}\vec{v} f_\alpha + \int d\vec{v}\,\underbrace{\vec{v}\frac{\vec{F}}{m_\alpha}\cdot\frac{\partial}{\partial\vec{v}} f_\alpha} = \frac{-1}{m_\alpha}\vec{R}_{\alpha\beta}$$

comes from $\int \vec{v} C d\vec{v}$ (where $\int d\vec{v} C$ is a statement of conservation of # density)

$$\vec{v}\frac{\vec{F}}{m_\alpha}\cdot\frac{\partial}{\partial\vec{v}} f_\alpha = -\left[\frac{\vec{F}}{m_\alpha}\cdot\underbrace{\frac{\partial}{\partial\vec{v}}\vec{v}}\right] f_\alpha$$

$$\frac{\partial}{\partial\vec{v}}\vec{v} = \hat{e}_i\hat{e}_j\frac{\partial}{\partial v_i}\cdot v_j$$

$$=\hat{e}_i\hat{e}_j\delta_{ij} = \overline{\overline{I}}$$

the identity tensor

- $\vec{R}_{\alpha\beta}$ = drag between $\alpha$ and $\beta$
- $\vec{R}_{\alpha\alpha} = 0$
  - i.e., no drag within a species

Recall: $\nu_{ei} = \dfrac{4\pi n_i Z^2 e^4}{m_e^2 v_e^2} \ln(\Lambda)$

Generally: $\vec{R}_{ei} = \nu_{ei} m_e n_e (\vec{U}_e - \vec{U}_i)$

$\vec{R}_{ie} = \nu_{ie} m_i n_i (\vec{U}_i - \vec{U}_e)$

$\Big\}\ \vec{R}_{ei} + \vec{R}_{ie} = 0$

$\Rightarrow$ total momentum of the plasma is conserved by collisions

$$\overbrace{\frac{\partial}{\partial t} n_\alpha \vec{U}_\alpha}^{\text{local momentum}} + \overbrace{\nabla \cdot \int d\vec{v}\,\vec{v}\vec{v} f_\alpha}^{\text{flux of momentum}} - \overbrace{\frac{q_\alpha}{m_\alpha} n_\alpha \left(\vec{E} + \frac{1}{c}\vec{U}_\alpha \times \vec{B}\right)}^{\text{change of momentum from } \vec{E}\text{-field}} = \underbrace{-\frac{1}{m_\alpha}\vec{R}_{\alpha\beta}}_{\text{momentum transfer to other species}}$$

$$= \int d\vec{v}\,\vec{v}\frac{\vec{E}}{m_\alpha}\cdot\frac{\partial}{\partial \vec{v}} f_\alpha$$

$\int d\vec{v} f_\alpha \vec{v} = n_\alpha \vec{U}_\alpha$

We can extract average velocity from $\nabla \cdot \int d\vec{v}\,\vec{v}\vec{v} f_\alpha$

Let $\vec{v} = \vec{U}_\alpha + \vec{v}'$

– note that $\int d\vec{v}'\,\vec{v}' f_\alpha = 0$

$\nabla \cdot \int d\vec{v}\,\vec{v}\vec{v} f_\alpha = \nabla \cdot (n_\alpha \vec{U}_\alpha \vec{U}_\alpha) + \nabla \cdot \int d\vec{v}'\,\vec{v}'\vec{v}' f_\alpha(\vec{v}')$

$\Rightarrow$ Define the <u>pressure tensor</u>

$\overline{\overline{P}}_\alpha = m_\alpha \int d\vec{v}'\,\vec{v}'\vec{v}' f_\alpha$

Then, rearranging the above equation

$$\frac{\partial}{\partial t} n_\alpha \vec{U}_\alpha + \nabla \cdot \int d\vec{v}\,\vec{v}\vec{v} f_\alpha = \frac{q_\alpha}{m_\alpha} n_\alpha \left(\vec{E} + \frac{1}{c}\vec{U}_\alpha \times \vec{B}\right) - \frac{1}{m_\alpha}\vec{R}_{\alpha\beta}$$

$$m_\alpha \Big(\frac{\partial}{\partial t} n_\alpha \vec{U}_\alpha + \underbrace{\nabla \cdot \int d\vec{v}\,\vec{v}\vec{v} f_\alpha}\Big) = q_\alpha n_\alpha \left(\vec{E} + \frac{1}{c}\vec{U}_\alpha \times \vec{B}\right) - \vec{R}_{\alpha\beta}$$

plug in value defined above

$$m_\alpha \Big[\frac{\partial}{\partial t} n_\alpha \vec{U}_\alpha + \nabla \cdot (n_\alpha \vec{U}_\alpha \vec{U}_\alpha) + \nabla \cdot \int d\vec{v}'\,\vec{v}'\vec{v}' f_\alpha(\vec{v}')\Big] = q_\alpha n_\alpha \left(\vec{E} + \frac{1}{c}\vec{U}_\alpha \times \vec{B}\right) - \vec{R}_{\alpha\beta}$$

$$m_\alpha \Big[\frac{\partial}{\partial t} n_\alpha \vec{U}_\alpha + \nabla \cdot (n_\alpha \vec{U}_\alpha \vec{U}_\alpha)\Big] + \nabla \cdot \underbrace{m_\alpha \int d\vec{v}'\,\vec{v}'\vec{v}' f_\alpha(\vec{v}')}_{\overline{\overline{P}}_\alpha} = q_\alpha n_\alpha \left(\vec{E} + \frac{1}{c}\vec{U}_\alpha \times \vec{B}\right) - \vec{R}_{\alpha\beta}$$

$$m_\alpha \Big[\frac{\partial}{\partial t} n_\alpha \vec{U}_\alpha + \nabla \cdot (n_\alpha \vec{U}_\alpha \vec{U}_\alpha)\Big] = q_\alpha n_\alpha \left(\vec{E} + \frac{1}{c}\vec{U}_\alpha \times \vec{B}\right) - \nabla \cdot \overline{\overline{P}}_\alpha - \vec{R}_{\alpha\beta}$$

use the continuity equation, $\frac{\partial}{\partial t} n_\alpha + \nabla \cdot n_\alpha \vec{v}_\alpha = 0$, to extract $n_\alpha$ from the momentum equation

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$$\Rightarrow n_\alpha m_\alpha \left[\frac{\partial}{\partial t}\vec{U}_\alpha + \vec{U}_\alpha \cdot \nabla \vec{U}_\alpha\right] = n_\alpha q_\alpha \left(\vec{E} + \frac{1}{c}\vec{U}_\alpha \times \vec{B}\right) - \nabla \cdot \overline{\overline{P}}_\alpha - \vec{R}_{\alpha\beta}$$

This is the <u>Fluid Momentum Equation</u>. Note that it is <u>not</u> complete since it depends on the unknown momentum tensor, $\overline{\overline{P}}_\alpha$

↳ need the next moment to get pressure tensor and so on

General Pattern:

0th

$$\frac{\partial}{\partial t} n_\alpha + \nabla \cdot n_\alpha \vec{U}_\alpha = 0$$

1st

$$\frac{\partial}{\partial t} m_\alpha n_\alpha \vec{U}_\alpha + (\quad) + \nabla \cdot \overline{\overline{P}}_\alpha = (\quad)$$

2nd

$$\frac{\partial}{\partial t} \overline{\overline{P}}_\alpha + (\quad) + \nabla \cdot [\text{third moment}] = (\quad)$$

– No closure of equations in a truly collisionless system – though, since we know collisions drive $f_\alpha$ toward a Maxwellian distribution, if the collisions are strong we can determine certain tensor properties to provide some closure

[to be addressed next time]