Lecture 5 - Collisions in the Boltzmann Equation 09/12/17

from last lecture...

Liouville's Equation

For a distribution function

$F(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_N, \vec{v}_N, t)$

$(\vec{x} = (x, y, z))$ $(\vec{v} = (v_x, v_y, v_z))$

that is constant along a trajectory in 6N space (Liouville's theorem):

$$\frac{\partial F}{\partial t} + \underbrace{\vec{v}_{6N} \cdot \nabla F}_{\text{"convection in the velocity of } F \text{ is } 0\text{"}} = \frac{\partial F}{\partial t} + \sum_s \vec{v}_s \cdot \frac{\partial}{\partial \vec{x}_s} F + \sum_s \frac{\vec{F}_s}{m} \cdot \frac{\partial}{\partial \vec{v}_s} F = 0$$

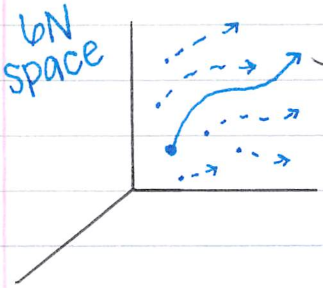
$\vec{F}_s = q(\vec{E}_s + \frac{1}{c}\vec{v}_s \times \vec{B}_s)$

⇒ Valid for both weakly and strongly correlated plasmas (Γ small or Γ large, respectively)

$$\vec{v}_{6N} = (\vec{v}_1, \underbrace{\vec{F}_1/m_1}_{\dot{\vec{v}}_1}, \ldots, \vec{v}_N, \vec{F}_N/m_N) \qquad \vec{v}_1 = \dot{\vec{x}}_1$$

↳ $\nabla_{6N} \cdot \vec{v}_{6N} = 0$ ⇒ Flow in 6N space is incompressible

6N space



- system trajector
  - an entire system can be described by a single point in 6N space and its trajectory
    - ↳ want to consider an ensemble of initial conditions/points

Collisionless Boltzmann Equation

→ describes the dynamics of a plasma in the weakly coupled regime in which $\Gamma \ll 1$

The Vlasov equation:

$f = f(\vec{x}, \vec{v}, t)$

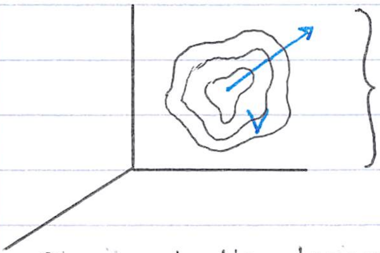
$$\frac{\partial f}{\partial t} + \vec{v}_{6N} \cdot \nabla f = 0$$

↳ $\vec{v}_6 = \left(\vec{v}, \frac{\vec{F}}{m}\right)$ } Movement in space is controlled by v and movement in v is controlled by F/m, $\vec{v}_6 = (v_x, F_x/m, v_y, F_y/m, v_z, F_z/m)$

$\nabla \cdot \vec{v}_6 = 0$

KLE

KLE



"Trajectory of the probability in time"
distribution vs. 1 point before

Similar to the derivation of Liouville's equation:

Use conservation of particles in 6D phase space

$$\frac{d}{dt}\int_V dV f = -\int_S d\vec{S}\cdot\vec{v}_6 f$$

$\vec{S}$ = surface in 6D space (5D)

– from the divergence theorem

$$\int_S d\vec{S}\cdot\vec{v}_6 f = \int_V dV\, \nabla_6\cdot\vec{v}_6 f$$

↓ plugging in ↓

$$\frac{d}{dt}\int_V dV f = -\left(\int_V dV\, \nabla_6\cdot\vec{v}_6 f\right) \rightarrow \int_V dV \underbrace{\left(\frac{\partial f}{\partial t} + \nabla_6\cdot\vec{v}_6 f\right)}_{\text{must} = 0} = 0$$

$$\frac{\partial f}{\partial t} + \underbrace{\nabla_6\cdot\vec{v}_6}_{\text{as before, } \nabla_6\cdot\vec{v}_6 = 0} f = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + \vec{v}_6\cdot\nabla_6 f = 0$$

Collisionless Boltzmann equation preserves entropy:

- entropy may be expressed as

$$S = -\int d\vec{x}\, d\vec{v}\, f \ln(f)$$

can be quite convoluted in phase space, so even small collisions will increase entropy

$\frac{\partial S}{\partial t} = 0$ – preservation statement

*Maximum S with respect to constraints gives a Maxwellian distribution

→

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Collisions in the Boltzmann Equation

{Chouduri Ch. 2, G-R Ch. 13}

In deriving the Boltzmann equation we have ignored collisions. If collisions are sufficiently weak this is justified.

↳ Under what conditions can collisions be neglected? ←

$$\frac{\partial f}{\partial t} \sim \underbrace{\vec{v} \cdot \nabla f}_{\text{convective derivative}} \sim \frac{\vec{E}}{m} \cdot \frac{\partial}{\partial \vec{v}} f$$

where $\nabla \sim 1/L$ — characteristic spatial scale length, $\partial/\partial v \sim 1/v_{th}$

$$\Rightarrow \frac{\partial}{\partial t} \sim \frac{v_t}{L}, \frac{F}{m v_t}$$

Want to compare these rates with typical scattering rates, ν

→ we can argue that if

$$\frac{v_t}{L}, \frac{F}{m v_t} \gg \nu$$

we can discard collisions

Following this,

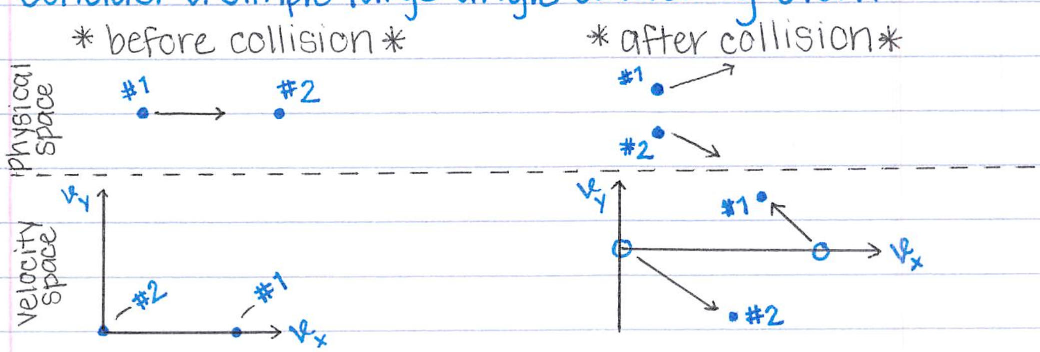
$\lambda = \frac{v_t}{\nu}$ = mean free path — the average distance travelled in a material between interactions [collisions]

if $L \ll \lambda$ we can ignore collisions

⇒ How do collisions act on particle distribution functions?

– how does it work, how do we describe it mathematically

Consider a simple large-angle scattering event:



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Collisional process moves particles in velocity space — will move/diffuse/distort the distribution function

$\Rightarrow$ f no longer constant along trajectory

However, collisions are usually small, not large angle, so they typically move in small steps in velocity space

NOW: Include collisions using the collision operator

$$\underbrace{\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f}_{\text{just free-streaming; no external force}} + \frac{\vec{F}_{ext}}{m}\cdot\frac{\partial}{\partial \vec{v}} f = \left(\frac{\partial f}{\partial t}\right)_c = C(f)$$

$\vec{F}_{ext}$: e.g., fields acting on particles (does not include drag force)

$C(f)$: just due to Coulomb interactions; does not include particle drag

$C(f)$ = The Collision Operator

- typically a non-linear integral operator
- non-local $\rightarrow$ includes the distribution function of the particles you are colliding with

Focus on Coulomb collisions:

1. Coulomb collisions conserve
   a) particle number
      $\int d\vec{v}\, C(f) = 0$
   b) momentum, when summed over species
      $\hookrightarrow$ linear & angular, depending on $\vec{F}_{ext}$
   c) energy, when summed over species
2. If f is in thermal equilibrium...

$$f = \frac{n_0}{(2\pi T/m)^{3/2}} e^{-(\frac{1}{2}mv^2 + e\varphi)/T}$$

in the form of a Maxwellian $\rightarrow$ equation of TE

... it should not change its distribution as a whole over time

$$C \rightarrow C(f) = 0, \quad \frac{\partial f}{\partial t} = 0$$

*If f is not originally in thermal equilibrium, collisions drive it to that state (Entropy increases until thermal equilibrium is reached ⇒ Boltzmann's number theorem)

③ Collisions act locally in physical/configuration space
⇒ λ_D is the cutoff for the interaction length for that specific collision (particles must be within λ_D of each other else they are shielded & cannot interact)

· What about velocity space?
↳ generally non-local in velocity space; particles with very different velocities can interact

How to go about describing the collision operator:
→ what form does it take? How do we construct it? ←

Fokker-Planck Equation for Collisions

Want to describe the evolution of f due to small-angle collisions.
└── this is really what we're interested in because, as found in lecture #3, small-angle collisions dominate by a factor of $\ln(\Lambda)$

*ignore space variations for now — e.g., spatial diffusion, viscosity

Define $\underbrace{P(\vec{v},\Delta\vec{v})}$
the probability a particle with velocity $\vec{v}$ in velocity space will suffer a jump $\Delta\vec{v}$ over a time Δt

Thus at a time t, the particle distribution function may be written as

$$f(\vec{x},\vec{v},t) = \int d\Delta\vec{v}\, \overbrace{f(\vec{v}-\Delta\vec{v}, t-\Delta t)}^{\text{f at an earlier time and velocity}}\, \underbrace{P(\vec{v}-\Delta\vec{v}, \Delta\vec{v})}_{\text{the probability of going from } f(\vec{v}-\Delta\vec{v},t-\Delta t)\to f(\vec{x},\vec{v},t)}$$

where as required,

$$\int d\Delta\vec{v}\, P(\vec{v},\Delta\vec{v}) = 1$$

(sum of all probabilities must = 1 by definition)

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Assume $\Delta\vec{v}$ to be small (w.r.t. $\vec{v}$) for small collisions

$\Delta\vec{v} \ll \vec{v}_{th,e}$

We can now expand the RHS of the particle distribution function about $\Delta\vec{v}$

> Math aside: The double-dot vector operator
>
> $$\vec{a}\vec{b}:\vec{c}\vec{d} = (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$
>
> —OR—
>
> $$(\vec{a}\vec{b}):(\vec{c}\vec{d}) = \vec{c}\cdot(\vec{a}\vec{b})\cdot\vec{d} = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})$$
>
> →Source: Wikipedia

$$f(\vec{v}-\Delta\vec{v}, t-\Delta t) = f(\vec{v},t) \overbrace{- \Delta t \frac{\partial}{\partial t} f}^{1^{st}\,\mathcal{O}\text{ expansion in time}} \overbrace{- \Delta\vec{v}\cdot\frac{\partial}{\partial\vec{v}} f(\vec{v},t)}^{1^{st}\,\mathcal{O}\text{ in } v} + \overbrace{\frac{1}{2}\underbrace{\Delta\vec{v}\Delta\vec{v}:\frac{\partial^2}{\partial\vec{v}\partial\vec{v}}}_{=\left(\Delta\vec{v}\cdot\frac{\partial}{\partial\vec{v}}\right)\left(\Delta\vec{v}\cdot\frac{\partial}{\partial\vec{v}}\right)} f}^{2^{nd}\,\mathcal{O}\text{ in } v}$$

$$P(\vec{v}-\Delta\vec{v}, \Delta\vec{v}) = P(\vec{v},\Delta\vec{v}) - \Delta\vec{v}\frac{\partial}{\partial\vec{v}}P + \frac{1}{2}\Delta\vec{v}\Delta\vec{v}:\frac{\partial^2}{\partial\vec{v}\partial\vec{v}}P$$

↓ put it all together ↓

$$\cancel{f(\vec{v},t)} = \Big(\overbrace{\cancel{f}}^{\text{move to LHS}} - \Delta t\frac{\partial f}{\partial t}\Big)\overbrace{\int d\Delta\vec{v}\, P(\vec{v},\Delta\vec{v})}^{=1\text{, as defined}} \quad \Big\}\ \text{terms independent of } \Delta v$$

expand in f ⌝ ⌜ expand in P

$$-\int d\Delta\vec{v}\,\Delta\vec{v}\cdot\left(\frac{\partial f}{\partial\vec{v}}P + \frac{\partial P}{\partial\vec{v}}f\right) \quad \Big\}\ \text{linear in } \Delta v$$

⌜ f is independent of Δv

$$+\frac{1}{2}\int d\Delta\vec{v}\,\Delta\vec{v}\Delta\vec{v}:\left[\frac{\partial^2 f}{\partial\vec{v}\partial\vec{v}}P + 2\frac{\partial f}{\partial\vec{v}}\frac{\partial P}{\partial\vec{v}} + f\frac{\partial^2 P}{\partial\vec{v}\partial\vec{v}}\right] \quad \Big\}\ \text{quadratic in } \Delta v$$

can pull out f) (P must stay

$$\Delta t\frac{\partial f}{\partial t} = \overbrace{-\frac{\partial}{\partial\vec{v}}\cdot\int d\Delta\vec{v}\,\Delta\vec{v}\, fP}^{\text{linear terms}} + \overbrace{\frac{1}{2}\frac{\partial^2}{\partial\vec{v}\partial\vec{v}}:\int d\Delta\vec{v}\,\Delta\vec{v}\,\Delta\vec{v}\, fP}^{\text{quadratic terms}}$$

($\frac{\partial}{\partial\vec{v}}$ can act directly on f, directly on P, or mix derivatives dividing through by Δt yields...

Fokker-Planck equation for collisions

*true for any small angle collisional process

$$\left(\frac{\partial f}{\partial t}\right)_c = \underbrace{-\frac{\partial}{\partial \vec{v}} \cdot \frac{d\langle \Delta \vec{v} \rangle}{dt} f}_{\text{drag term}} + \underbrace{\frac{1}{2} \frac{\partial^2}{\partial \vec{v} \partial \vec{v}} : \frac{d\langle \Delta \vec{v} \Delta \vec{v} \rangle}{dt} f}_{\text{diffusion term}}$$

~ i.e., diffusion of the distribution in velocity space

the two distinct terms/processes associated with collisions

Define averages:

$$\frac{d}{dt}\langle \Delta \vec{v} \rangle = \frac{1}{\Delta t} \int d\Delta \vec{v} \, P \Delta \vec{v}$$

$$\frac{d}{dt}\langle \Delta \vec{v} \Delta \vec{v} \rangle = \frac{1}{\Delta t} \int d\Delta \vec{v} \, \Delta \vec{v} \Delta \vec{v} \, P(\vec{v}, \Delta t)$$

*note that these are all functions of $\vec{v}$

What is the vector direction of $\frac{d}{dt}\langle \Delta \vec{v} \rangle$, the drag term?

↳ must be opposite the velocity vector

$$\frac{d}{dt}\langle \Delta \vec{v} \rangle \sim -\nu \vec{v}$$

Recall that earlier we found ν for e^--i collisions ↴

$$\frac{d}{dt}\langle \Delta \vec{v} \rangle_{ei} = -\frac{4\pi n_i Z^2 e^4 \ln(\Lambda)}{m^2 v^3} \vec{v}$$

Returning to the Fokker-Planck equation

$$\frac{\partial}{\partial t} f = -\frac{\partial}{\partial \vec{v}} \left(\frac{d}{dt}\langle \Delta \vec{v} \rangle\right) f + (\text{diffusion stuff})$$

where $f = f(\vec{v}, t)$

How do we calculate drag on this distribution function?

(how the mean velocity of the distribution function evolves in time)

- must first multiply through by $\vec{v}$
  - otherwise the integral would just give the number density, which is constant

$$\frac{\partial}{\partial t} f(\vec{v},t)\cdot\vec{v} = -\frac{\partial}{\partial \vec{v}}\left(\frac{d}{dt}\langle\Delta v\rangle\right) f(\vec{v},t)\cdot\vec{v}$$

• integrate with respect to $\vec{v}$

diffusion term drops out

$$\frac{\partial}{\partial t}\underbrace{\int d\vec{v}\, f(\vec{v},t)\vec{v}}_{n_0\vec{U}} = -\int d\vec{v}\,\vec{v}\,\underbrace{\frac{\partial}{\partial\vec{v}}\cdot\frac{d}{dt}\langle\Delta v\rangle}_{\text{how drag/mean drift changes with time}} f(\vec{v},t)$$

background density

$$\frac{\partial}{\partial t}(n_0\vec{U}) = -\nu\underbrace{\int d\vec{v}\,\vec{v}\, f(\vec{v},t)}_{n_0\vec{U}}$$

pull out

mean drift velocity of the plasma ($=\vec{v}_0$)

$$\Rightarrow n_0\frac{\partial\vec{U}}{\partial t} = \underbrace{-\nu n_0\vec{U}}$$

Coulomb collisions shown to cause slowing down of the distribution

- diffusion shows temperature changes with collisions

** The Fokker-Planck equation is the generic form of the Collision operator in an ionized plasma

↳ no neutral collisions

We can also use...

<u>Landau form of the Collision Operator</u>

- looking at the evolution of a species α:

$$\frac{\partial}{\partial t}f^\alpha + \vec{v}\cdot\nabla f^\alpha + \frac{\vec{F}}{m_\alpha}\frac{\partial}{\partial v}f^\alpha = \sum_\beta \underbrace{C(f^\alpha, f^\beta)}$$

distribution function of species α

rate of change of f^α due to collisions with the species β

$C(f^\alpha, f^\beta)$ = the Landau operator, a function of v

* this form introduces nonlinearity in velocity space

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$$C(f^\alpha, f^\beta) = -\frac{\partial}{\partial \vec{v}} \cdot \left\{ \frac{2\pi q_\alpha^2 q_\beta^2 \ln(\Lambda)}{m_\alpha} \right.$$

identity tensor

$$\left. \cdot \int d\vec{v}' \frac{(u^2 \overline{\overline{I}} - \vec{u}\vec{u})}{u^3} \cdot \left[\frac{1}{m_\beta} f^\alpha(\vec{v}) \frac{\partial}{\partial \vec{v}'} f^\beta(\vec{v}') - \frac{1}{m_\alpha} f^\beta(\vec{v}') \frac{\partial}{\partial \vec{v}} f^\alpha(\vec{v}) \right] \right\}$$

$d\vec{v}'$ will be integrated out

this operator also describes scattering, where the particle is scattered around an angle (= 0 if in the direction of $\vec{v}$)

where $\vec{u} = \vec{v} - \vec{v}'$;

All terms dependent on differentiation in velocity

(as expected from the F-P averages definitions)

⇒ You get both drag terms (1st order) and diffusion terms (2nd order) out of this Landau operator

If you put in a Maxwellian in the limit where $\Delta T_{\alpha\beta} = 0$, $C \to 0$:

f_α, f_β have Maxwellian distribution with same temp., T

$$\frac{\partial}{\partial \vec{v}} f^\alpha = -f^\alpha \frac{m_\alpha \vec{v}}{T}$$

masses cancel out

$$-\int d\vec{v}' \frac{(u^2 \overline{\overline{I}} - \vec{u}\vec{u})}{u^3} \left[\frac{1}{\cancel{m_\beta}} f^\alpha(\vec{v}) \left(\frac{-f^\beta(\vec{v}') \cancel{m_\beta} \vec{v}'}{T} \right) - \frac{1}{\cancel{m_\alpha}} f^\beta(\vec{v}') \left(\frac{-f^\alpha(\vec{v}) \cancel{m_\alpha} \vec{v}}{T} \right) \right]$$

pull T's out

$$\frac{\partial}{\partial \vec{v}} f^\alpha = -\int d\vec{v}' \boxed{\frac{(u^2 \overline{\overline{I}} - \vec{u}\vec{u})}{T u^3} \cdot \vec{u}} f^\alpha f^\beta$$

$$(u^2 \overline{\overline{I}} - \vec{u}\vec{u}) \cdot \vec{u} = u^2 \vec{u} - \vec{u} u^2 = 0$$

$$\frac{\partial}{\partial \vec{v}} f^\alpha = 0$$

⇒ Collisions have no effect when you have reached thermal equilibrium!

→

for electrons scattering off of ions:

$$\left(\frac{\partial f}{\partial t}\right)_{ei} = \frac{2\pi n_i Z^2 e^4 \ln(\Lambda)}{m_e^2} \frac{\partial}{\partial \vec{v}} \frac{(\overline{\overline{I}} v^2 - \vec{v}\vec{v})}{v^3} \cdot \underbrace{\frac{\partial}{\partial \vec{v}} f^e}_{= \left(\frac{\partial}{\partial v^2} f^e\right) 2\vec{v}}$$

which gives 0 when dotted into

This tells how the electron distribution function changes with time due to collisions with ions

└ ignoring e-e collisions but retaining e-i collisions = Lorentz gas

⇒ Does not change energy but does change angle!
This operator only scatters the pitch angle of the particle; no energy scattering

{ this is the simplest form used in homework #2 }

$$\frac{\partial}{\partial \vec{v}} \cdot \frac{(\overline{\overline{I}} v^2 - \vec{v}\vec{v})}{v^3} \cdot \frac{\partial}{\partial \vec{v}} f^e(v^2) = 0$$

→ solution is an arbitrary function of v^2