Lecture 4 - Energy Exchange & Liouville's Equation 09/07/17

from last time...

Fractional Ionization in Dilute Plasmas → Coronal equilibrium

$$\frac{n_p}{n_H} = \frac{\langle \sigma_i v \rangle}{\langle \sigma_R v \rangle} = \text{coronal equilibrium} \rightarrow \text{a function of } T_e \text{ only}$$

σ_i: ionization cross-section

σ_R: radiative recombination cross-section

Electron-Ion Collision Rate

→ momentum scattering

$$\frac{d}{dt} v_z = -\nu_{ei} v_z$$

$$\nu_{ei} = \frac{4\pi n_i Z^2 e^4}{m_e^2 v_e^3} \underbrace{\ln(\Lambda)}_{\text{typically} \sim 20}$$

$$\Lambda \equiv \frac{\lambda_D}{b_0} = \text{a ratio of max to min impact parameters} = 4\pi n \lambda_D^3$$

We move on to other collisions now —

Electron-Electron Collisions (similar to electron-ion)

→ Use the reduced mass (masses now comparable) $\mu = m_e/2$

Rate (# encounters/time):

$$\nu_{ee} \sim \frac{\nu_{ei} n_e}{Z^2 n_i}$$

*energy exchange between electrons (will return to this)

Ion-Ion Collisions

What dominates momentum scattering of ions? — other ions

↳ use the reduced mass, $\mu = m_i/2$

Rate:

$$\nu_{ii} \simeq \frac{16\pi n_i Z^4 e^4}{m_i^2 v_i^3} \ln(\Lambda)$$

Scales like...

$T_i \simeq T_e$ (per previous temperature uniformity assumption)

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and since our focus is on mostly thermal motion — $T \sim mv^2$

$\hookrightarrow T_i \sim m_i v_i^2 \sim m_e v_e^2 \sim T_e$

$$\Rightarrow \nu_{ii} \sim \frac{16\pi n_i Z^4 e^4}{m_i^{1/2} T_i^{3/2}} \ln(\Lambda) \sim \nu_{ei}\left(\frac{m_e}{m_i}\right)^{1/2}$$

Energy Exchange

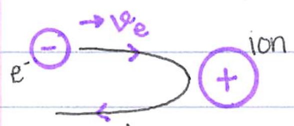
→ So far we've looked at momentum exchange — now energy!

During $e^- - e^-$ collisions the change in energy can be of order unity so electrons equalize their energy at a rate given by $\nu_{ee} \sim \nu_{ei}$

$\hookrightarrow$ same for ion-ion exchange; rate $\sim \nu_{ii} \sim \sqrt{m_e/m_i}\,\nu_{ei}$

How long does it take for ions & electrons to exchange energy?

Consider an electron colliding with an ion:



large angle scattering event

"Rough argument": $\Delta v_e \sim \underbrace{2v_e}_{\text{sped up by large angle of scatter}}$

Since momentum must be conserved

$m_e \Delta v_e \sim m_i \Delta v_i$

$\hookrightarrow 2 v_e m_e \sim m_i \Delta v_i$

$$\Rightarrow \Delta v_i \sim \frac{2 v_e m_e}{m_i}$$

In general it is true that

$$\langle v_i^2 \rangle \sim \frac{\Delta v_i^2}{\tau} t$$

t — time over which v_i^2 is averaged

τ — characteristic timescale of the system

τ for our system $= 1/\nu_{ei}$ [time/# encounters]

$$\rightarrow \langle v_i^2 \rangle \sim \frac{v_e^2 m_e^2}{m_i^2} \nu_{ei} \cdot t$$

$$m_i \langle v_i^2 \rangle \sim \frac{v_e^2 m_e^2}{m_i} \nu_{ei} \cdot t \sim m_e v_e^2$$

using the momentum conservation argument above

↓ dividing through by $m_e v_e^2$ ↓

$$\frac{m_e}{m_i} \nu_{ei} \cdot t \sim 1$$

$$\nu_{E_{ei}} = \frac{1}{t}\left(\frac{m_e}{m_i} \nu_{ei} \cdot t\right)$$

$$\nu_{E_{ei}} \sim \nu_{ei} \underbrace{\frac{m_e}{m_i}}_{\text{very small value}}$$

⇒ energy exchange is the slowest process

What about exchange for electron-ion collision with the electrons and ions at different temperatures?

→ assuming e^- hotter than ions

$$\underbrace{m_i \langle v_i^2 \rangle}_{\sim T_i} \sim \frac{v_e^2 m_e^2}{m_i} \nu_{ei} \cdot t \sim \underbrace{m_e v_e^2}_{\sim T_e}$$

So the energy exchange rate

$$\nu_{E_{ei}} \sim \nu_{ei} \frac{m_e}{m_i}$$

is also the rate at which ion temperature increases.

Next topic: How do particles evolve in space?

– ignore/forget collisions; add them back later

Liouville's Equation {Ref: Chouduri Ch. 1, G-R Ch. 22}

We would like to construct a set of equations that can be used to explore the dynamics of a plasma

– system consists of N particles

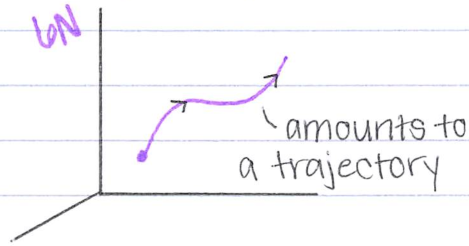
– complete state of a system is given by 6N variables

$$(\vec{x}_1, \vec{v}_1, \vec{x}_2, \vec{v}_2, \ldots, \vec{x}_N, \vec{v}_N)$$

└ $\vec{x} = (x, y, z)$ └ $\vec{v} = (v_x, v_y, v_z)$

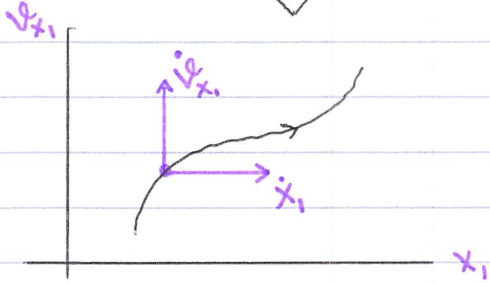
Define some 6N dimensional space

↳ the behavior of the system is given by a trajectory in 6N space



⇨ trajectory can be projected onto 2D space

⇩



Generally, we would want to consider a statistical description represented by an <u>ensemble</u> of systems with the same mean properties.

The local probability of finding the system in a state is:

$$\underbrace{dP}_{\text{probability of finding the system in that small (local) element}} = F(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_n, \vec{v}_n, t)\, \overbrace{d\vec{x}_1 d\vec{v}_1 \ldots d\vec{x}_n d\vec{v}_n}^{\text{volume element}}$$

(probability; F: distribution function)

where

$\int dP = 1$ (closed system)

Liouville's Theorem: F is a constant if you move along a trajectory defined by a member of the ensemble

→ i.e., F is a constant along a trajectory in 6N space (in the absence of collisions)

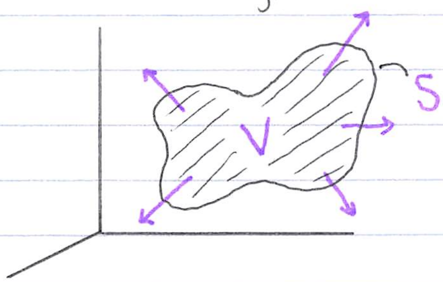
To show that Liouville's Theorem is correct, we need to use the continuity equation for particles:

Consider a volume V in 6N phase space

$$\frac{\partial}{\partial t}\int_V dV\, F = -\int_S F\, d\vec{V}_{6N}\cdot d\vec{S}$$

surface integral

where $\vec{V}_{6N}$ is the 6N velocity
$\vec{V}_{6N} \equiv (\dot{\vec{x}}_1, \dot{\vec{v}}_1, \dots, \dot{\vec{x}}_N, \dot{\vec{v}}_N)$



Use the divergence theorem:

$$\int_V dV \left(\frac{\partial F}{\partial t} + \nabla\cdot(\vec{V}_{6N} F) \right) = 0$$

This is true for any V, so we may say

$$\frac{\partial F}{\partial t} + \sum_s \underbrace{\frac{\partial}{\partial \vec{x}_s}\cdot(\dot{\vec{x}}_s F)} + \sum_s \underbrace{\frac{\partial}{\partial \vec{v}_s}\cdot(\dot{\vec{v}}_s F)} = 0$$

$\dot{\vec{v}}_s = \frac{1}{m}\vec{F}_s$

$$= \vec{v}_s\cdot\nabla F + F\frac{\partial}{\partial \vec{x}_s}\cdot\vec{v}_s$$

$$= \vec{v}_s\cdot\frac{\partial}{\partial \vec{x}_s}F + F\cancel{\frac{\partial}{\partial \vec{x}_s}\cdot\vec{v}_s}^{\,0}$$

since $\vec{x}_s, \vec{v}_s$ are taken to be independent variables

$$= \frac{1}{m}\vec{F}_s\cdot\frac{\partial}{\partial \vec{v}_s}F + \frac{F}{m}\frac{\partial}{\partial \vec{v}_s}\cdot\vec{F}_s$$

$\vec{F}_s = q\left(\vec{E}_s + \frac{1}{c}\vec{v}_s\times\vec{B}_s\right)$

$$\rightarrow \frac{\partial}{\partial \vec{v}_s}\cdot\vec{F}_s = \frac{q}{c}\underbrace{\frac{\partial}{\partial \vec{v}_s}\cdot(\vec{v}_s\times\vec{B}_s)}$$

$$\frac{\partial}{\partial \vec{v}}(\vec{v}\times\vec{B}) = \cancel{\frac{\partial}{\partial v_x}(v_y B_z - v_z B_y)}^{\,0} + \dots = 0$$

Putting this all together gives us <u>Liouville's Equation</u>:

$$* \ \frac{\partial F}{\partial t} + \sum_s \vec{v}_s\cdot\frac{\partial}{\partial \vec{x}_s}F + \sum_s \frac{\vec{F}_s}{m}\cdot\frac{\partial}{\partial \vec{v}_s}F = 0 \ *$$

Note that: $\nabla_{6N}\cdot\vec{V}_{6N} = 0$

$\Rightarrow$ Flow in 6N space is incompressible

In a Hamiltonian system now:

$$\dot{p}_i = \frac{-\partial \mathcal{H}}{\partial q_i}, \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

$$\nabla_{6N} \cdot \vec{v}_{6N} \sim \frac{\partial}{\partial p_i} \cdot \dot{p}_i + \frac{\partial}{\partial q_i} \dot{q}_i$$

$$\sim -\frac{\partial^2 \mathcal{H}}{\partial q_i \partial p_i} + \frac{\partial^2 \mathcal{H}}{\partial p_i \partial q_i} = 0$$

This allows us to write the Liouville's equation using canonical variables; e.g., cylindrical or spherical coordinates

*The Liouville equation is an essentially exact equation, but it contains too much information. It is valid for Γ large or small but → we will be most interested in the limit $\Gamma \ll 1$ where individual particles are only weakly correlated ←

In the limit where particles are uncorrelated:

$$F(\vec{x}_1, \vec{v}_1, \ldots, \vec{x}_N, \vec{v}_N, t) = F_1(\vec{x}_1, \vec{v}_1, t) F_1(\vec{x}_2, \vec{v}_2, t) \ldots F_1(\vec{x}_N, \vec{v}_N, t)$$

↳ the particle distribution function can be written as the product of single-particle pdfs

Single-particle particle distribution function:

- define $f(\vec{x}, \vec{v}, t)$, related to the 6D single-particle particle distribution function $F_1(\vec{x}, \vec{v}, t)$

$$f(\vec{x}, \vec{v}, t) = N F_1(\vec{x}, \vec{v}, t)$$

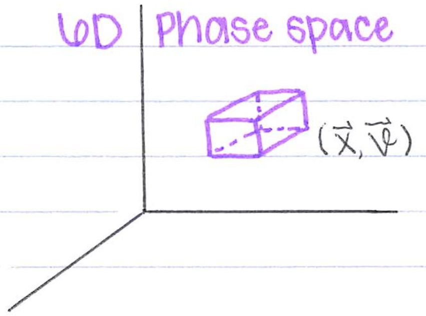
where $\int_V d^3x\, d^3v\, F_1 = 1$ (F_1 describes all 6D space)

↳ The equation for $f(\vec{x}, \vec{v}, t)$ is the Vlasov equation

The Vlasov Equation [Collisionless Boltzmann equation]

Want to obtain an equation for $f(\vec{x}, \vec{v}, t)$ such that we are able to describe the dynamics of a plasma in the weakly coupled regime in which $\Gamma \ll 1$.

KLE



$dN = f(\vec{x}, \vec{v}, t)\, d\vec{x}\, d\vec{v}$
$= \#$ of particles in a volume $d\vec{x}\, d\vec{v}$

We can derive an equation for f by integrating the Liouville equation over $d^3x_2 d^3v_2 \ldots d^3x_N d^3v_N$

Want to show:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} f + \frac{q}{m}\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) \cdot \frac{\partial}{\partial \vec{v}} f = 0$$

*note that in this description $\vec{x}, \vec{v}, t$ are independent variables
→ no collisions
→ no spontaneous radiation
→ no synchrotron/Bremsstrahlung radiation