Lecture 3 - Ionization & Collisions

09/05/17

Recap from lecture 2...

Ionization Equations

→ Saha equation ←

ionization potential [13.6 eV]

$$\underbrace{\frac{1}{n a_B}}_{\text{small}}\left(\frac{T}{4\pi E_H}\right)^{3/2} e^{-E_H/T} = \frac{\delta_e^2}{1-\delta_e}$$

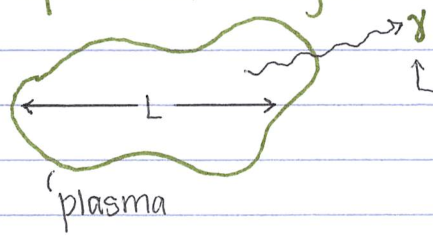
where δ_e = fraction of ionization

$$= \frac{n_e}{n_H + n_e} = \frac{n_e}{n}$$

Predicts:

- significant ionization even for $T \ll E_H$
- not valid for most plasma systems
- okay for use in the core of stars
- requires perfect thermal equilibrium, even for photons
  - ↳ requires that the photons have a blackbody spectrum

*This is not the case in most plasmas because the mean-free-path is too long



photons escape before equilibrating

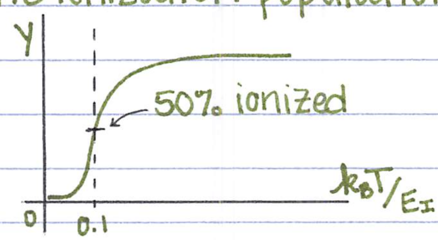
From ASTR601 – Radiative Processes:

{Taught by Cole Miller; Used "Radiative Processes in Astrophysics" by Rybicki & Lightman}

The Saha equation assumes equilibrium and no other sources of ionization to determine ionization populations

↳ for ionization fraction y

If k_BT is even $0.1 \times E_I$, then the ionization fraction is 50%!



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*Assumes solely radiative thermal (from γ's) ionization processes, i.e., a perfect Blackbody is required
↳ when the density is high enough to infringe on the atoms, thermal ionization isn't as relevant as pressure ionization

For Hydrogen

cgs units

$$\frac{y^2}{1-y} = \underbrace{\frac{4\times10^{-9}}{\rho}}_{\text{spatial volume}} \underbrace{T^{3/2}}_{\text{momentum volume}} \underbrace{e^{-1.6\times10^5/T}}_{\text{Boltzmann factor requirement (gives distribution)}} \quad , \quad y = \text{ionization fraction} = \frac{n^+}{n} = \frac{n_e}{n}$$

⇒ Smaller ρ yields larger ionization fraction, y

- For a plasma of fixed temperature but varying density, the Saha equation implies that the degree of ionization goes up when the density goes down
  ↳ the ionization rate per volume, therefore, just goes like the number density

HOWEVER — recombination rate involves two particles and hence must go as the product of their densities
↳ recombination rate increases more rapidly when the density goes up, and the resulting ionization fraction goes down

For an ionization pair:
- $1/\rho \propto \Delta x$, spatial volume per pair
- $\Delta p \propto T^{3/2}$, momentum volume per pair
  (where $v_{th} = \sqrt{3k_BT/2m}$

$\Rightarrow \Delta x \Delta p$ = phase space volume
└ want to give the pair a larger volume to explore (more favorable condition for ionization)

There is a Saha equation connecting any two successive stages of ionization →

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U_j & U_{j+1} are the corresponding partition functions

$$\frac{N_{j+1}N_e}{N_j} = \frac{2U_{j+1}(T)}{U_j(T)}\left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{\underbrace{-\chi_{j,j+1}}/k_BT}$$

N = total # densities

$\chi_{j,j+1}$ = ionization potential

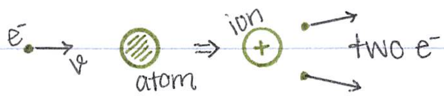
where j and j+1 refer to stages of ionization

Coronal Equilibrium Model

Derived for the solar corona (after the Saha eqn. did not work)

Dynamical processes that control ionization = reaction processes

→ Collisional ionization



Rate of change of electron # density

$$\dot{n}_e = \nu_I n_H$$

rate of ionization

σ = cross-section ~ area

$$\nu_I = \underbrace{\langle \sigma n v \rangle_v}_{\langle\,\rangle_v = \text{ average over velocity}} = \text{\# of ionizations/second}$$

ν_I may be written as an integral over velocity space

distribution function of e^- as a function of $\vec{v}$

$$\langle \sigma n v \rangle = \underbrace{\int d\vec{v}\, f_e(\vec{x},\vec{v})}_{\text{units of density}}\sigma v$$

What about the cross-section of the ionization?

→ a property of the interaction

→ dependent on velocity

E = energy of electrons

$$\sigma_I = 4\pi a_B^2 \left(\frac{E_H}{E}\right)^2 \underbrace{\left(\frac{E}{E_H} - 1\right)}_{E > E_H \Rightarrow \text{ the threshold to ionize}}$$

Behavior @ large E: if the particles are moving too fast, the interaction with the ions is reduced

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*for e^- velocity very large, cannot give enough energy to ionize

We can then calculate the change in electron # density

$$\dot{n}_e = n_H n_e \underbrace{\langle \sigma_I v \rangle}$$

divide $\int d\vec{v} f_e$ by e

only a function of T, not the density

$\langle \sigma_I v \rangle = \langle \sigma_I v \rangle (T_e)$

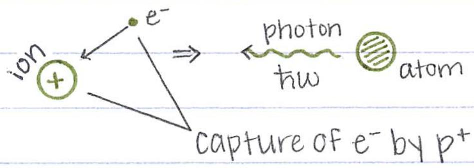
NOW: Recombination

which when in balance gives coronal equilibrium

*For low density, radiative recombination dominates

(three-body recombination has low probability unless the density is very high)

→ Radiative Recombination



Including recombination, the total $\dot{n}_e$ becomes

proton density

$$\dot{n}_e = \frac{d}{dt} n_e = n_H n_e \langle \sigma_I v \rangle - n_e n_p \langle \sigma_R v \rangle$$

σ_R = recombination cross-section

$= \sigma_R(T_e)$, as with σ_I

Reorganizing...

$$\dot{n}_e = n_e [n_H \langle \sigma_I v \rangle - n_p \langle \sigma_R v \rangle]$$

$= 0$ for steady-state/equilibrium conditions

it's not true thermal equilibrium, but reasonable for a macro system where these time scales are short (plasma is in equilibrium, photons are not)

$n_e \neq 0$ so $[n_H \langle \sigma_I v \rangle - n_p \langle \sigma_R v \rangle]$ must $= 0$

$$\Rightarrow \frac{n_p}{n_H} = \frac{\langle \sigma_I v \rangle}{\langle \sigma_R v \rangle} = \text{Coronal Equilibrium}$$

a function of T_e only

→ Ionization around 1eV

*see Fig. 10.5 at the end of this lecture – good for a lot of systems that we're interested in

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Collision Rates in Fully Ionized Plasma {Ref: G&R Ch. 11}

First, consider how electrons scatter from ions

- when an electron collides with an ion, the electron is gradually deflected by the long-range Coulomb field of the ion

→ Inspect momentum & energy transfer and scattering, assuming a completely ionized plasma

(ignore collisions between neutrals)

Simple-Minded Estimate for Collision Rates

How close must e^- get to the ion for the Coulomb repulsion to equal the thermal energy of the electron?

$m_e v^2 \sim \frac{e^2}{r}$ required

thermal & potential energies must be comparable

$$\longrightarrow r = \frac{e^2}{m_e v^2}$$

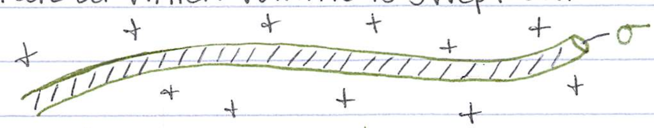
Expressing physical cross-section...

$$\sigma \sim \pi r^2$$

$$= \pi \frac{e^4}{m_e^2 v^4}$$

↓ plug into collision rate expression ↓

$\nu_{ei} \sim n \underbrace{\sigma v}_{\text{rate at which volume is swept out}}$ = # of encounters w/ an ion / time



$$\sim n\pi \frac{e^4}{m_e^2 v^4} v = \frac{n\pi e^4}{m_e^2 v^3}$$

higher particle velocity → smaller collision rate

This describes very close encounters/large-angle collisions

BUT! This is not the dominant process!

↳ Because of the long-range nature of the Coulomb force, small-angle collisions are much more frequent than large-angle ones

The cumulative effect of many small-angle collisions/

deflections turns out to be larger than the effect of relatively fewer large-angle collisions / deflections

*results in a logarithmic scaling factor for small r-values

But how big is this rate?

$$\nu_{ei} = \frac{n\pi e^4}{m_e^2 v^3} = \frac{n\pi e^4}{m_e^{1/2} T^{3/2}}$$

$\frac{1}{2} m v_{th}^2 \equiv T_e$

→ Write this with respect to the Debye length

k_D = the Debye screening wave vector

$$k_D^2 = \frac{1}{\lambda_D^2} = \frac{4\pi n e^2}{T_e}$$

$$\hookrightarrow T_e = \underbrace{4\pi n e^2}\lambda_D^2$$

recall $\omega_{pe}^2 = \frac{4\pi n e^2}{m_e}$ the plasma frequency

$$\frac{\nu_{ei}}{\omega_{pe}} = \frac{\pi n \cancel{e^4}}{\cancel{m_e^{1/2}}(4\pi n \cancel{e^2}\lambda_D^2)^{3/2}} \frac{\cancel{m_e^{1/2}}}{(4\pi n \cancel{e^2})^{1/2}}$$

$$\sim \underbrace{\frac{1}{n\lambda_D^3}} \ll 1$$ for all weakly-coupled plasmas

because this is small, we know that plasma waves are not affected by collisions here

*collisions are weak at least on the plasma freq. time scale

Consider now a more careful discussion of collisional processes in which we will calculate the rates of several processes:

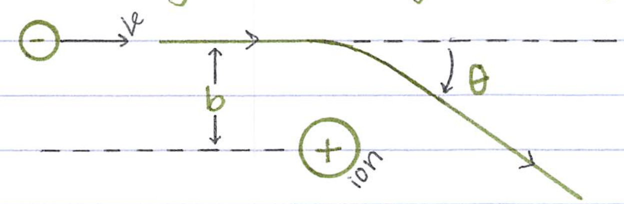
- electron momentum scattering
- ion momentum scattering
- electron-ion energy exchange

⇒ But first we need to find our small-angle ν_{ei}

→

Electrons Scattering Off of Ions

Ions of charge Ze (not just H anymore)



b = impact parameter

Deflection in Center of Mass frame:

- a particle acted upon by an inverse-square-law force will execute a hyperbolic orbit given by

$$\tan\left(\frac{\theta}{2}\right) = \frac{\overbrace{Ze^2}^{=(Ze)(e)}}{\mu b v^2}$$

reduced mass $1/\mu = 1/m_i + 1/m_e \simeq \underbrace{1/m_e}$

$\mu \sim m_e$ for electron-ion interaction

for large-angle scattering...

→ 90° scattering

$$b = \frac{Ze^2}{mv^2} \equiv b_0$$ intrinsic scale size in scattering processes

exactly what we found before for r from our simple-minded estimate

from this

$$\sigma_0 = \pi b_0^2$$

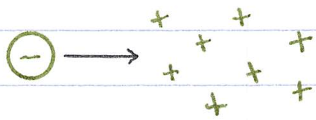
intrinsic CS for large-angle scattering

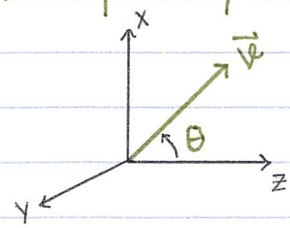
$$= \underbrace{\frac{\pi Z^2 e^4}{m_e^2 v^4}}$$

same as before, just generalized to include ions other than H

NOW: Look at small-angle deflection

Want to explore the contributions due to many interactions that deflect the electrons by many small angles





$\langle \Delta v_x \rangle = 0$, $\langle \Delta v_x^2 \rangle \neq 0$

small changes will cancel out when space-averaged

$\langle \Delta v_y \rangle = 0$, $\langle \Delta v_y^2 \rangle \neq 0$ → there will be some mean v^2 deflection

$\langle \Delta v_z \rangle \neq 0$ (will be slowed)

$\langle \Delta v_\perp^2 \rangle = \langle \Delta v_x^2 \rangle + \langle \Delta v_y^2 \rangle$

$= v^2 \sin^2(\theta)$

$\sim v^2\theta^2$ → small-angle approximation

* $\Delta v_\perp^2$ & θ will increase with time as we get more and more small deflections

⇒ Random walk in θ and $v_\perp$

A single scattering with impact parameter b will scatter through

$$\tan\left(\frac{\theta_b}{2}\right) \sim \frac{\theta_b}{2} = \frac{b_0}{b}$$

w.r.t. large-angle intrinsic values

Let's consider a large # of scatterings...

For N scatterings:

$\theta = \sum_i \theta_{bi}$

$\theta_{bi} = \pm\theta_b \pm \delta$ → θ_{bi} can be $<\theta_b$ or $>\theta_b$ (it's random)

↳ $\langle \theta \rangle = \sum_i \theta_{bi} = 0$

want to find the time-dependence of this

$\langle \theta^2 \rangle = \langle \sum_i \theta_{bi} \sum_j \theta_{bj} \rangle$

i^{th} & j^{th} events uncorrelated

⇒ only get non-zero value for $\langle\theta^2\rangle$ when $i=j$

$= \sum_i \theta_{bi}^2$

$= \theta_b^2 \sum_i$ sum over # scatters

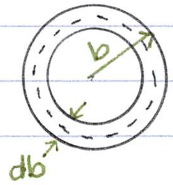
$\langle\theta^2\rangle = N\theta_b^2$ $\Rightarrow$ need to know how N is related to time

Over a time Δt we have

$\Delta N = \underbrace{d\sigma_b}\, n_i v \Delta t$ = # of small-angle scattering events

$d\sigma_b = 2\pi b\, db$

└ the cross-section for a ring of radius b and thickness db



Must add up all scatterings due to b as b varies

(must consider all b-values!)

Thus, over time Δt, when we inspect how change in N impacts $\langle\theta^2\rangle$, we find

$$d\langle\theta^2\rangle = \Delta N \theta_b^2$$

$$= 2\pi b\, db\, n_i v \Delta t \underbrace{\theta_b^2}_{\theta_b = 2 b_0/b}$$

$$= 2\pi b\, db\, n_i v \Delta t\, 4 b_0^2/b^2$$

→ divide through by Δt to get a rate

$$\frac{d\langle\theta^2\rangle}{dt} = d\sigma_b n_i v \theta_b^2$$

$$= 8\pi n_i v b_0^2 \underbrace{\frac{db}{b}}$$

for a given db

if you integrate over db, you get a divergence!

↓ integrate over b ↓

$$\frac{d\langle\theta^2\rangle}{dt} = n_i v \int_{b_0}^{\lambda_D} \frac{db}{b}\, 8\pi b_0^2$$

outside λ_D the electrons will shielded out, so this must be b_{max} (electrons don't feel ion charge beyond λ_D)

small θ means b_0 is the lower limit

*Note: sensitivity to bounds is a logarithm, so you can be a bit sloppy

$$= n_i v 8\pi b_0^2 \ln\left(\frac{\lambda_D}{b_0}\right)$$

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Note that the integral diverges logarithmically as $b \to \infty$
$\Rightarrow$ small-angle scattering dominates ($\lambda_D \gg b_0$)

Returning to $v_\perp$:

$$\frac{d\langle \Delta v_\perp^2 \rangle}{dt} = v^2 \frac{d\langle \theta \rangle^2}{dt^2}$$

$$= v\, 8\pi n_i v b_0^2 \ln(\lambda_D / b_0)$$

but we want the electron slowing rate, which is related to v_z

- Assume e^- not losing energy (energy conservation)
  - $\hookrightarrow v_\perp^2 + v_z^2 = \text{constant}$ — i.e. momentum scattering

$$\Delta v_\perp^2 + 2 v_z \underbrace{\Delta v_z}_{} = 0$$

assumed Δv_z small w.r.t. v_z because of small θ scatter

$$\langle \Delta v_z \rangle = -\frac{\langle \Delta v_\perp^2 \rangle}{2 v_z} = -\frac{\langle \Delta v_\perp^2 \rangle}{2 v}$$

only $\langle \Delta v_z \rangle$ was non-zero

$$\frac{d\langle \Delta v_z \rangle}{dt} = -\frac{1}{\cancel{2}}\frac{1}{\cancel{v}} \cancel{8}^{4}\pi n_i \cancel{v^3}^{2} b_0^2 \ln\left(\frac{\lambda_D}{b_0}\right)$$

$\downarrow$ plug in b_0 $\downarrow$

$$\underbrace{\frac{d\langle \Delta v_z \rangle}{dt}}_{\text{want this to } = \nu_{ei} v} = -\frac{4\pi n_i Z^2 e^4}{m_e^2 \cancel{v^4}^{2}} \cancel{v} \ln\left(\frac{\lambda_D}{b_0}\right)$$

We can now find ν_{ei} = # collisions/time, small θ

$$* \; \nu_{ei} = \frac{4\pi n_i Z^2 e^4}{m_e^2 v^3} \ln(\Lambda)$$

$\hookrightarrow \Lambda \equiv \dfrac{\lambda_D}{b_0}$

*note that $\nu_{ei} \propto \frac{1}{v^3} \to$ higher energy particles suffer fewer collisions

$\Rightarrow$ Same as found before w/ large θ, but with a factor of $\ln(\Lambda)$ - the factor that shows small θ dominates

$$b_0 = \frac{Ze^2}{\underbrace{m_e v^2}_{\text{like } v_{th}^2 = 2T_e/m_e}} = \frac{Z^2 e^2}{T_e}$$

$$\sim \frac{e^2 \, 4\pi n}{T \;\; 4\pi n} = \frac{1}{\lambda_D^2}\frac{1}{4\pi n}$$

$\hookrightarrow 1/\lambda_D^2$

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$$\Rightarrow \Lambda = \frac{\lambda_o}{b_o} = \lambda_o \lambda_D^2 4\pi n$$

$$= \underbrace{4\pi \lambda_D^3 n}_{} \gg 1$$

tabulated in plasma formulary

-just plug in T_e

Although Λ depends on n & T, its logarithm is fairy insensitive to the exact values of these parameters.

- for the most part we can say $\ln(\Lambda) \sim 20$, but a few more specific values are shown below

	$n(m^{-3})$	$T(eV)$	$\ln(\Lambda)$
Solar wind	10^7	10	26
Van Allen belts	10^9	10^2	26
Earth's ionosphere	10^{11}	10^{-1}	14
Solar corona	10^{13}	10^2	21
Gas discharge	10^{16}	10^0	12
Process plasma	10^{18}	10^2	15
Fusion experiment	10^{19}	10^3	17
Fusion reactor	10^{20}	10^4	18

{G&R Table 11.1}

*It is evident that $\ln(\Lambda)$ varies by not more than a factor of two as the plasma parameters range over many orders of magnitude

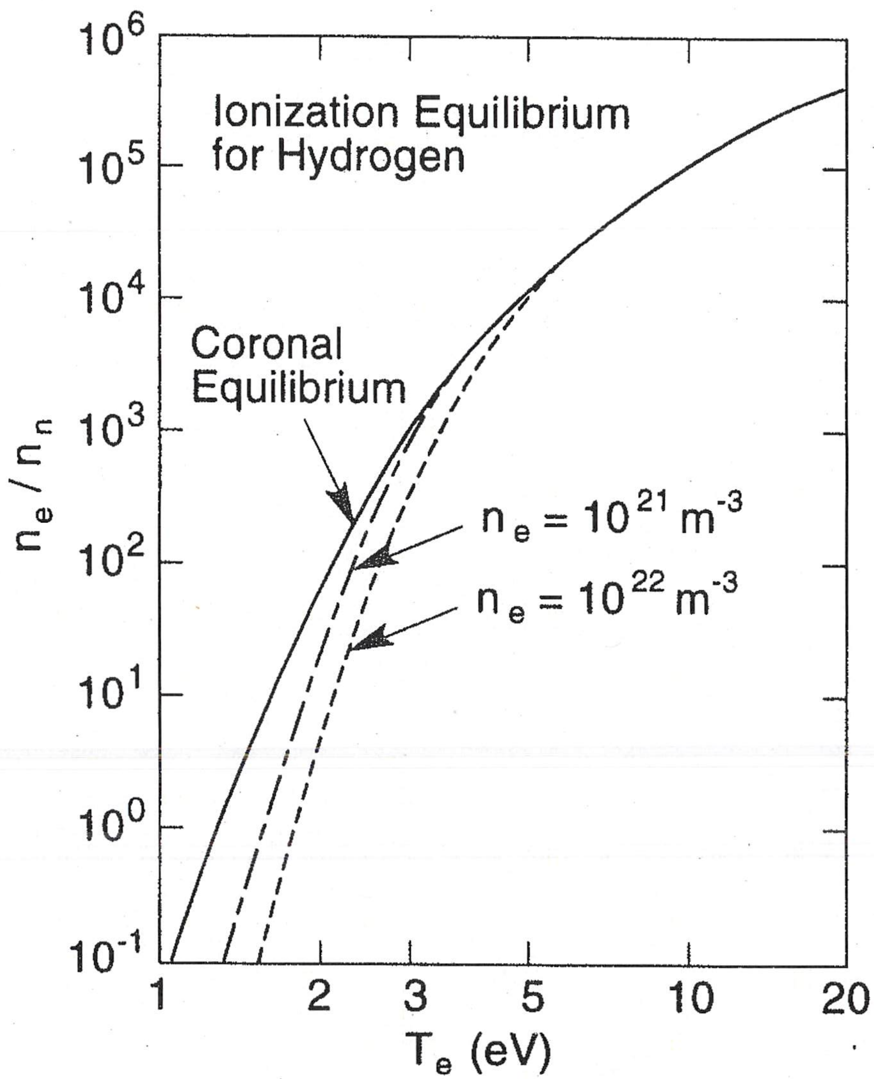

Figure 10.5. Ionization equilibrium for hydrogen in the coronal equilibrium model, and at higher electron densities with three-body recombination included.

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