# Lecture #2 - Shielding, Waves, and Ionization 08/31/17

Important parts from last time...

## Plasma Parameter

$$\Gamma = \frac{U_c}{U_T} \sim \frac{e^2}{r_s^{1/3} T}$$

↑ $r_s = 1/n^{1/3}$, $n$=# density

- $\Gamma < 1$: nearby particles weakly interacting
  - ↳ thermal motion *focus is here*
- $\Gamma > 1$: nearby particles strongly interacting (liquids, crystals)
  - ↳ Coulomb motion

The plasma parameter may also be expressed in terms of the Debye length:

$$\Gamma = \frac{1}{4\pi}\frac{1}{(\underbrace{n\lambda_D^3}_{})^{3/2}}$$

↑ # of particles in Debye sphere

$\lambda_D$ = Debye length

→ a screening length

It is also useful to consider the Debye length in terms of temp.

$$\lambda_D^2 \equiv \frac{T}{4\pi n e^2}$$

↖ can check here that yes, $\lambda_D$ is a length dimensionally (T in eV)

## Distribution Functions

→ Maxwell-Boltzmann Distribution ←

$\sim Ce^{-E/T}$, C=constant

↑ gives probability of finding a particle w/ energy E

where $E = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) + q\phi$

→ Particle Distribution Function ←

We can describe the distribution of particle velocities in a plasma by the particle distribution function

$$f_{e,i}(\overbrace{\vec{x}, \vec{v}, t}^{7\text{ arguments}})$$

x, y, z ↗ ↖ $v_x, v_y, v_z$

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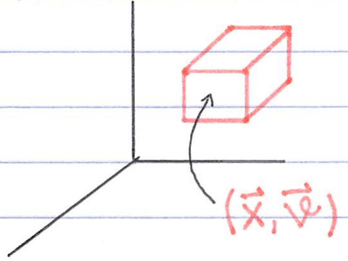
This is a 6-dimensional space $\vec{x}, \vec{v}$



a small cube with
6-dimensional volume

This infinitesimal volume may be written:

$$d\vec{x}\,d\vec{v} = d^3x\,d^3v = dx\,dy\,dz\,dv_x\,dv_y\,dv_z$$

Then,

$$dN = f(\vec{x},\vec{v})\,d^3x\,d^3v$$

└ # of particles in infinitesimal volume

Integrating over all of phase space to obtain the total # of particles ↴

$$N_{e,i} = \int d^3x \underbrace{\int d^3v\, f_{e,i}(\vec{x},\vec{v})}_{n_{e,i}(\vec{x}) = \text{local number density}}$$

$$N_{e,i} = \int d^3x\, n_{e,i}(\vec{x})$$

In thermal equilibrium, the particle distribution function takes the form

$$f = \frac{n_0}{(2\pi T/m)^{3/2}}\, e^{-(\frac{1}{2}mv^2 + q\varphi)/T}$$

*we note that the normalization of the phase space

$$\int_{-\infty}^{\infty} dp \frac{1}{\sqrt{\pi}} e^{-p^2} = 1$$

yields a local number density:

$$n = \int d^3v\, f = n_0 e^{-q\varphi/T}$$

## Debye Shielding

Consider an electron-ion plasma which is $\underbrace{\text{charge-neutral}}$

Equal # of charges such that net = 0 →

$$-n_e q_e = n_i q_i$$

How does the plasma respond to a test charge, $q_t$?

↳ this will tell us about how the charges respond to each other

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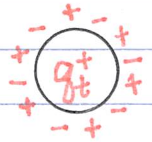
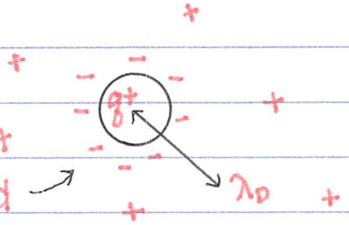


→ charges will move such that negatives are more inward →



The plasma immediately tries to shield out the positive test charge with a cloud of negative charges — this leads to a scale length $\lambda_D$ for shielding

↳ that is, cancelling out the $\vec{E}$-field of $q_t^+$

We want to quantify this: $\vec{E} = -\nabla\varphi$

$$\nabla\cdot\vec{E} = -\nabla^2\varphi = 4\pi\rho$$

test charge location

$$\rho = n_e q_e + n_i q_i + q_t\,\delta^3(\vec{x}-\vec{x}_t)$$

- $n_e q_e = n_{e,0}\, q_e\, e^{-q_e\varphi/T}$ , $\uparrow n_e$
- $n_i q_i = \underbrace{n_{i,0}}_{\text{initial local \# densities}}\, q_i\, \underbrace{e^{-q_i\varphi/T}}_{\text{take these values to be small}}$ , $\downarrow n_i$

shielding is increasing the local # density of electrons and decreasing the local # density of ions

↳ $\Gamma \ll 1$; weakly coupled

$\Rightarrow \left|\frac{q\varphi}{T}\right| \ll 1$

expand ↓

- $n_e q_e \sim n_{e,0}\, q_e\left(1 - q_e\varphi/T\right)$
- $n_i q_i \sim n_{i,0}\, q_i\left(1 - q_i\varphi/T\right)$

↓ plugging back into $-\nabla^2\varphi = 4\pi\rho$ ↓

$$-\nabla^2\varphi = 4\pi q_i n_{i0}\left(1 - q_i\varphi/T\right) + 4\pi q_e n_{e0}\left(1 - q_e\varphi/T\right) + 4\pi q_t\,\delta^3(\vec{x}-\vec{x}_t)$$

BUT: initially charge neutral, so

$$-n_{e0}q_e = n_{i0}q_i$$

$$\hookrightarrow -\nabla^2\varphi = -\underbrace{\sum_{e,i}\frac{4\pi q_{e,i}^2 n_{e,i,0}}{T}}_{\equiv k_D^2}\varphi + 4\pi q_t\,\delta^3(\vec{x}-\vec{x}_t)$$

$\equiv k_D^2$, the characteristic length scale
[dimensions of $1/L^2$]

$$\Rightarrow -\nabla^2\varphi + k_D^2\varphi = 4\pi q_t\,\delta^3(\vec{x}-\vec{x}_t)$$

a differential equation with source term

→

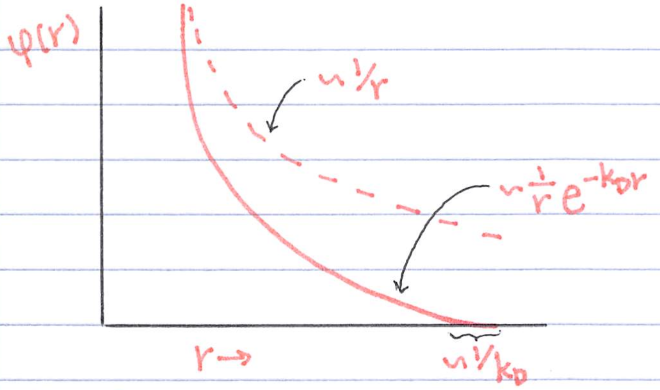
Assume a spherically symmetric system with $q_t$ @ center

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\varphi\right)+k_D^2\varphi = 4\pi q_t\frac{2\delta(r)}{4\pi r^2}$$

**Solving this in first homework**

Solution:

$$\varphi(r) = q_t\frac{e^{-k_D r}}{r}$$



⇒ The test charge is shielded out in a distance $r \sim 1/k_D \equiv \lambda_D$

Therefore,

$$n\lambda_D^3 = 1/\gamma^{3/2} \underbrace{\gg 1}_{\text{a large \# of shielding electrons required}}$$

From all this, we may also say

- → the distribution method (f) chosen was okay
- → this does not work if $n\lambda_D^3 \sim 1$

Example of some numbers: Solar Wind

dist = 1 AU, $T_{e,i} \sim 10$ eV, $n \sim 1$ cm$^{-3}$

$$\lambda_D = \frac{1}{\sqrt{4\pi e^2}}\sqrt{\frac{T}{n}} = \frac{7.43\times10^2\cdot T^{1/2}}{n^{1/2}}\ \text{cm}$$

$$= \frac{7.43\times10^2\cdot\sqrt{10}}{1}\ \text{cm}$$

⇒ $\lambda_D \sim 2\times10^3$ cm

$n\lambda_D^3 \sim 10^{10}$ electrons

→

## Plasma Waves

A plasma can support many types of collective oscillations

e.g. sound waves as in normal gas

Can also support unusual waves involving only the motion of electrons

↳ electrostatic plasma waves
→ electromagnetic plasma waves
→ longitudinal waves

} these have high frequency with small ion response

Let

$$\vec{E} = \text{Re}\{\underbrace{\hat{x}}_{\text{field in x-direction}}\overset{\text{complex amplitude}}{E_0}e^{-i\omega t}\}$$

* Note: No $\vec{B}$-field; 1D system

Electron motion along $\vec{E}$

$\vec{v} = \text{Re}\{\hat{x}v_0 e^{i\omega t}\}$

↳ a longitudinal wave

(!) Ignore the ion response — they're too heavy to respond

Applying Newton's law

$$m_e\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\vec{v} = -e\vec{E}$$

→ This is a small amplitude wave — for very small amplitude, terms quadratic in the amplitude will be so small as to be negligible, so we neglect them

$$\vec{v}\cdot\nabla\vec{v} \to 0$$

$\Rightarrow -i\omega m_e v_0 = -eE_0$

↓ plug into Maxwell ↓

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}E$$

$\vec{J} = \text{Re}\{-n_0 e v_0 e^{-i\omega t}\hat{x}\}$

↳ project onto x ⤵

→

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$J_x = -n_0 e v_0$

$$\underbrace{\hat{x}\cdot\nabla\times\vec{B}}_{=0\ (\text{no } \vec{B}\text{-field})} = \frac{4\pi}{c}J_x + \frac{1}{c}\frac{\partial}{\partial t}E_x$$

$\frac{\partial}{\partial t}\rightarrow i\omega$

$$0 = \frac{4\pi}{c}(+n_0 e v_0) + \frac{i\omega}{c}E_0$$

$v_0 = \frac{eE_0}{i\omega m_e}$ from Newton's Laws

$$0 = \frac{4\pi}{c}n_0 e\left[\frac{eE_0}{i\omega m_e}\right] + \frac{i\omega}{c}E_0$$

$$0 = \frac{i\omega}{c}E_0\left\{1 - \frac{4\pi n_0 e^2}{\omega^2 m_e}\right\}$$

This bracket needs to be zero... ...therefore this must be like $\omega^2$; in fact it is $\omega_{pe}^2$

$\Rightarrow \omega_{pe}^2 = \frac{4\pi n e^2}{m_e}$ The Plasma Frequency

Aside: Was it okay that we neglected the ions?

- Would have been:

$$0 = \frac{i\omega}{c}E_0\left\{1 - \frac{4\pi n_0 e^2}{\omega^2 m_i}\right\} \rightarrow \omega_{pi}^2 = \frac{4\pi n_0 e^2}{m_i} \ll \omega_{pe}^2$$

- This is reconfirmed since $\omega_{pi}^2/\omega_{pe}^2$ yields $\sim m_e/m_i$, which is TINY

$$E_0\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right) = 0$$

since $E_0 \neq 0$, this forces $\omega = \omega_{pe}$

- We can also define from this

$$\varepsilon \equiv 1 - \frac{\omega_{pe}^2}{\omega^2} = \text{dielectric constant for high frequencies}$$

$E_0(\varepsilon(k,\omega))$

**this type of wave occurs as a result of some kind of charge separation in the plasma

→ for $\omega = \omega_{pe}$ (as above), $\varepsilon = 0$

Let's return to the nonlinearity & thermal motion we threw away... Why, and was it really okay to do so?

Non-Linearity: for no $B_0$, $\nabla\times\vec{E}=0$

└→ electrostatic wave

$\vec{E} = -\nabla\varphi$, $\frac{\partial}{\partial x}\neq 0$

with $i\vec{k}\rightarrow ik_x$ ⌐ 1D wave vector for this electrostatic wave

↙ we left this out before

$\vec{E}=\text{Re}\{\hat{x}E_{00}e^{ik_x x - i\omega t}\}$

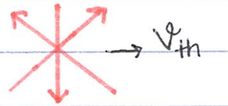
→ Require oscillation in x be small compared to the wavelength

$kx_0 \ll 1$

└ $e^-$ amplitude of oscillation

$x_0 \sim \frac{v_0}{\omega}$ so: $\underbrace{\frac{kv_0}{\omega}\ll 1}_{\text{amplitude not too large}}$

*too big and things start to break down BUT we also cannot do small amplitude stuff & still drop the nonlinear terms like we did earlier

Thermal Motion: 

$T\equiv\frac{1}{2}mv_{th}^2$

To ignore this, we want

$$\underbrace{\frac{v_{th}}{\omega}}_{\text{thermal electron displacement}} \ll \frac{1}{k} \rightarrow \frac{kv_{th}}{\omega}\ll 1$$

$$\frac{kv_{th}}{\omega}\sim k\boxed{\frac{v_{th}}{\omega_{pe}}}\sim k\left[\frac{2v_{th}^2 m_e}{(4\pi ne^2)}\right]^{1/2}$$

$\frac{v_{th}}{\omega_{pe}}=\lambda_D \Rightarrow$ want $k\lambda_D\ll 1$ to ignore thermal motion contributions

* if you do actually account for thermal motion, you find

$$\omega^2 = \omega_{pe}^2 + \frac{1}{2} B k^2 v_{th}^2$$

## Ionization

{Book by H.R. Griem "Principles of Plasma Spectroscopy" is a good reference}

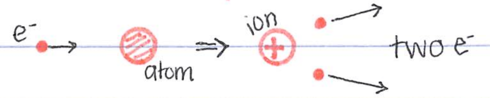
Under what conditions can a neutral gas become ionized?

→ Start with a neutral gas, heat it up (supply energy), and particles will become ionized — we'd like to know...

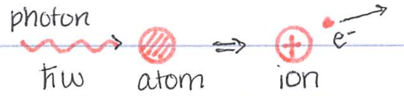
- How do particles become ionized?

Two dominant processes:

① Electron Impact Ionization

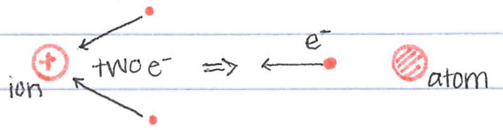


② Photo-ionization

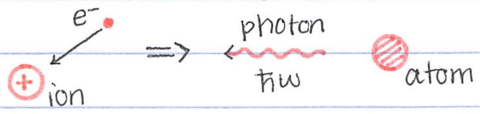


You also have the inverse processes:

① Three-Body Recombination



② Radiative Recombination



* Three-Body recombination is only important in very high density plasmas — otherwise it's negligible compared with Electron Impact, Photo-ionization, and Radiative Recombination

In thermal equilibrium these processes balance and determine the fraction of atoms that are ionized.

→ Consider hydrogen (H) for simplicity

$n_p$ = density of protons

$n_H$ = density of H (atoms)

$$f = \frac{n_p}{n_p + n_H} = f(n_t, T)$$

$n_t = n_H + n_p$

*assume $H_2$ disassociated

Let $P(N_e, N)$ be the probability that there will be $N_e$ electrons in a gas of N total particles

$$P = \frac{Z_e^{N_e}}{N_e!} \cdot \frac{Z_p^{N_p}}{N_p!} \cdot \frac{Z_H^{N_H}}{N_H!}$$

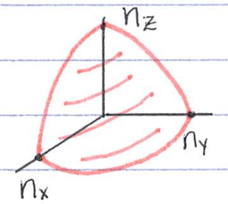
$N_H + N_p = N_t$

- partition function:

$$Z_e = \sum_E e^{-E/T}$$

*Note that this is the classical limit of Fermi-Dirac statistics and is only valid for $n a_B^3 \ll 1$

Bohr radius



$$E_n = \frac{\hbar^2}{2m}\left(\frac{n^2\pi^2}{L^2}\right), \quad n^2 = n_x^2 + n_y^2 + n_z^2$$

quantized states

$$= \frac{g_e}{8} \int_0^\infty 4\pi n^2 dn\, e^{-\frac{\hbar^2}{2m} \cdot \frac{n^2\pi^2}{L^2} \cdot \frac{1}{T}}$$

8 instead of 16 because we only want $n_x, n_y, n_z = 0$

$$= \frac{g_e}{2} \pi \left(\frac{2mL^2T}{\pi^2\hbar^2}\right)^{3/2} \underbrace{\int_0^\infty dp\, p^2 e^{-p^2}}_{= \sqrt{\pi}/4}$$

$$Z_e = g_e \left(\frac{m_e L^2 T}{2\pi\hbar^2}\right)^{3/2}$$

$$Z_p = g_p \left(\frac{m_p L^2 T}{2\pi\hbar^2}\right)^{3/2}$$

$$Z_H = g_H \left(\frac{m_p L^2 T}{2\pi\hbar^2}\right)^{3/2} e^{+E_H/T}$$

Only the ground state: When fraction of ionization is of order unity.

$E_H \equiv 13.6$ eV $\gg T$

↳ will be in ground state

Want to maximize $\ln(P)$ with respect to $N_e$

→ $N_e = N_p$, $N_H = N - N_e$

*recalling that $\ln(n!) \simeq n\ln(n) - n$

$$\frac{\partial}{\partial N_e}\ln(P) = \ln(Z_e)+\ln(Z_p)-\ln(Z_H)-\ln(N_e)-\ln(N_e)+\ln(N-N_e)=0$$

$$\Rightarrow \frac{Z_e\cdot Z_p}{Z_H} = \frac{N_e^2}{N-N_e}$$

↓plugging in↓

$$g_e\left(\frac{m_eL^2T}{2\pi\hbar^2}\right)^{3/2} g_p \cancel{\left(\frac{m_pL^2T}{2\pi\hbar^2}\right)^{3/2}} \frac{1}{g_H}\cancel{\left(\frac{2\pi\hbar^2}{m_pL^2T}\right)^{3/2}} e^{-E_H/T} = \frac{N_e^2}{N-N_e}$$

$$\text{spin degeneracy}\begin{cases} g_e = g_p = 2 \\ g_H = 4 \end{cases}$$

$$\left(\frac{m_eL^2T}{2\pi\hbar^2}\right)^{3/2} e^{-E_H/T} = \underbrace{\frac{N_e^2}{N-N_e}}_{\text{for any } n_i,\ n_i = N_i/L^3\ ;\ L^3 = \text{volume of gas}}$$

$$\Rightarrow \left(\frac{m_e\cancel{L^2}T}{2\pi\hbar^2}\right)^{3/2} e^{-E_H/T} = \frac{n_e^2}{n-n_e}\cancel{L^3}$$

Use the Bohr radius to get rid of $\hbar$:

$$a_B = \frac{\hbar^2}{e^2 m}$$

↳ $\hbar^2 = e^2 m a_B$

$$E_H = \frac{e^2}{2a_B} = 13.6\text{ eV}$$ only the ground state is accessible

↳ $e^2 = 2a_BE_H$

$$\hbar^2 = 2ma_B^2E_H$$

↓plugging in↓

$$\frac{1}{n}\left(\frac{\cancel{m_e}T}{2\pi(2\cancel{m}a_B^2E_H)}\right)^{3/2} e^{-E_H/T} = \underbrace{\left(\frac{n_e^2}{n-n_e}\right)\frac{1}{n}}_{\frac{n_e}{n} = \delta_e = \text{fractional ionization}}$$

*$n = n_e + n_H$

→

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⇒ The Saha Equation

$$\underbrace{\frac{1}{na_B}}_{\ll 1\ (\#\text{ in the deBroglie }\lambda\text{ sphere})}\left(\frac{T}{4\pi E_H}\right)^{3/2} e^{-E_H/T} = \frac{\delta_e^2}{1-\delta_e}, \quad \underbrace{E_H = 13.6\text{ eV} \quad a_B = \frac{\hbar^2}{me^2} = 5.3\times10^{-9}\text{cm}}_{\text{with these and T, we can find } \delta_e}$$

- Valid under equilibrium conditions
  - ↳ accounts for ALL ionization processes
  - ↳ doesn't describe most plasmas of interest...

Trends:
- as $T \to 0$, $\delta_e \to 0$
- as $T \to$ large, $\delta_e \to 1$

* what T gives $\delta_e \sim 1$?

$\frac{E_H}{T} \gg 1$ or $T \ll E_H$

ex: for $\delta_e = 1/2$

$$\left(\frac{T}{4\pi E_H}\right)^{3/2} e^{-E_H/T} = \frac{1}{2}\underbrace{na_B^3}_{\ll 1}$$

$\rightarrow E_H/T \gg 1$

$T \ll E_H$

⇒ free electrons have many available states