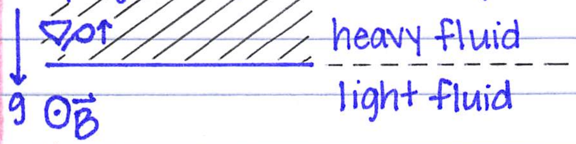
Lecture 23 - Double Adiabatic Equations

11/14/17

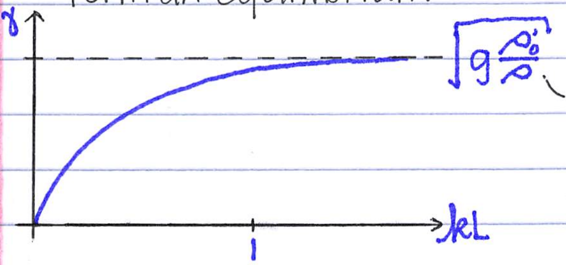
from last time...

Rayleigh-Taylor Instability



R-T instability is the instability of the interface between two fluids of different densities, arranged heavy over light

*Note: $\vec{B}$ may be varied to form an equilibrium



characteristic growth

where γ is the growth rate of the R-T instability

*$\rho_0' = \frac{\partial}{\partial y}\rho_0$

⇒ B is convected with the flow

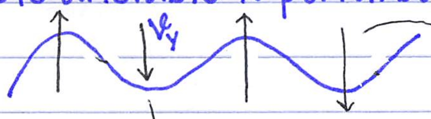
⇒ Nearly incompressible for $\omega_f \gg \gamma$

characteristic fast mode frequency

$$\omega_f = \left(\sqrt{C_S^2 + C_A^2}\right)k$$

want to get rid of this in order to extract the slow time variation of Rayleigh-Taylor

This is unstable to perturbations like



direction of perturbation

$\odot \vec{B}$ does nothing to stop this and is convected along

From this system we have a perturbed momentum eq.

$$\rho_0 \frac{\partial}{\partial t}\vec{u}_1 = -\nabla\underbrace{\left(p_1 + \frac{\vec{B}_0\cdot\vec{B}_1}{4\pi}\right)}_{} - \rho_1 g\hat{y}$$

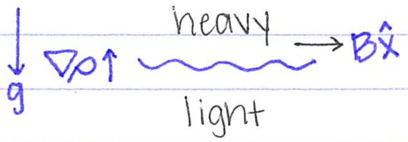
Big terms; together, the "gradient term"

P & B terms associated with fast modes

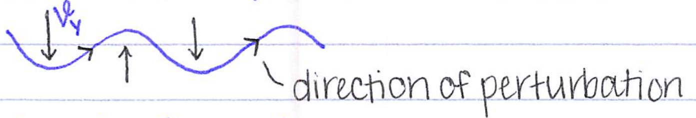
- R-T is controlled by the gravity term
  - ↳ Take the curl of the perturbed momentum equation to eliminate the gradient term

(resulting equation lacks pressure and magnetic perturbations)

What if $\vec{B}$ is in a different direction?



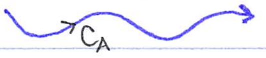
⇒ The instability will perturb $\vec{B}$ like



→ This will produce a magnetic tension force that can potentially stabilize this system

This forms an Alfvèn wave!

└ under what condition does this kill the R-T instability?



$$\underbrace{\omega_A}_{\text{Stabilizes when}} \sim kC_A$$

$$\omega_A > \sqrt{\frac{g}{L}} - \gamma_g$$

*this is highly dependent on the value of k, but in general this is the condition for stabilization

For any fixed $\vec{B}$:

- If $\vec{k} \perp \vec{B}$ there can be no stabilization
- For a $\vec{B}$-field that is twisting in space, you can always find a condition to stabilize
  - └ find $\vec{k} \parallel \vec{B}$ to find the stabilization criteria (can be done analytically via differential equations)

Our MHD equations were derived assuming the collisions were large enough that pressure was a constant scalar property

↳ What if there are pressure anisotropies?

Double Adiabatic / CGL Model

The Double Adiabatic / CGL model is used if the collisions are not strong enough to make the pressure tension isotropic.

(the CGL model is sometimes useful if a full kinetic treatment is too complex)

- Use J, μ invariance / adiabatic condition to find a set of equations for weak collisions ⇒ CGL Equations

(Chew, Goldberger, and Low (1956)

Starting with the collisionless Boltzmann...

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \underbrace{\frac{q}{m}\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial}{\partial\vec{v}}}f = 0$$

As with our MHD equations, we want to order such that the electric & magnetic forces are large

↳ this term dominant when considering ordering

- To lowest order:

$$\frac{q}{m}\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial}{\partial\vec{v}}f_0 = 0 \qquad \text{Eq. ①}$$

f_0 = equilibrium distribution function

Where $E_{\parallel}=0$ assumed

(because this component can't be transformed away)

→ Move to the $\vec{E}\times\vec{B}$ drift frame to eliminate $\vec{E}$ above

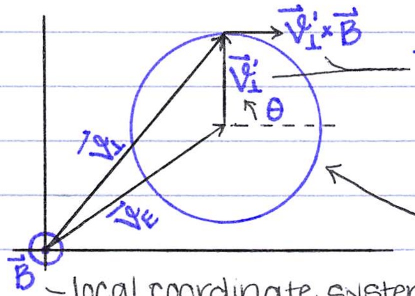
$$\vec{v}' = \vec{v} - \underbrace{\frac{c}{B^2}\vec{E}\times\vec{B}}_{\vec{v}_E}$$

thus Eq. ① becomes

$$\frac{q}{m}\frac{1}{c}(\vec{v}'\times\vec{B})\cdot\frac{\partial}{\partial\vec{v}}f^{(0)} = 0$$

- In the $\vec{v}\cdot\vec{B}=0$ plane:

*pressure in the perpendicular plane is the same everywhere by setup definition



$\vec{v}_\perp'$ = perpendicular velocity in $\vec{E}\times\vec{B}$ frame (to lowest order just rotates around)

$f^{(0)}$ is constant along this circle of constant $\vec{v}_\perp'$ (circle is path of the particle around the field)

– local coordinate system with $\vec{B}$ out of page

⇒ $f^{(0)}$ is independent of the phase angle, θ

$$f^{(0)}(\vec{x},\vec{v},t) = f^{(0)}(\vec{v}_\perp', v_\parallel, \vec{x}, t)$$

What does this independence imply?

In a local magnetic coordinate system, the parallel pressure decouples from the transverse pressure

└ i.e. $P_\parallel$ does not have to be the same as $P_\perp$ everywhere

↖ the pressure in the direction along the local $\vec{B}$-field

$$P_\parallel = m\int d\vec{v}\, f^{(0)}(v_\parallel - \langle v_\parallel \rangle)^2$$

- where the perpendicular component is a function of θ

$$\left.\begin{aligned} P_{\perp_1} &= m\int d\vec{v}\, f^{(0)} v_\perp'^2 \underbrace{\cos^2(\theta)}_{avg = 1/2} \\ P_{\perp_2} &= m\int d\vec{v}\, f^{(0)} v_\perp'^2 \sin^2(\theta) \end{aligned}\right\}$$

$P_{\perp_1}$ and $P_{\perp_2}$ are the same when averaged over one period. The average of $\sin^2$ & $\cos^2$ both = ½

→ $P_{\perp_1} = P_{\perp_2} = P_\perp$

We can write P as a tensor:

$$\overline{\overline{P}} = \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{pmatrix}$$

– the pressure tensor for a system where $P_\parallel \neq P_\perp$ like it previously did for an isotropic system

which in vector form may be expressed as

$$\Rightarrow \overline{\overline{P}} = P_\parallel \hat{b}\hat{b} + P_\perp(\underset{\text{identity tensor}}{\overline{\overline{I}}} - \hat{b}\hat{b})$$

*recall $\hat{b} \equiv \frac{\vec{B}}{|\vec{B}|}$ from Lec#16

*If we can determine $P_\parallel$ and $P_\perp$, we can construct a set of closed fluid equations. We can use the adiabatic variables to do this.

μ Conservation

magnetic moment – an adiabatic invariant

$\mu \sim \frac{v_\perp^2}{B} \sim$ constant , known

↓ average over velocity space ↓

$\frac{\langle v_\perp^2 \rangle}{B} \sim$ constant

*also an average over all particles

KLE

Since we express T as an energy, and our numerator is really mv^2,
we may substitute $T_\perp$ for $\langle v_\perp^2\rangle$

↑Ref: L'16

$\underbrace{T_\perp \text{ for } \langle v_\perp^2\rangle}_{\text{they're "essentially the same"}}$

insert ↘

$$\rightarrow \frac{\langle v_\perp^2\rangle}{B} \sim \frac{T_\perp}{B}\left(\frac{\rho}{\rho}\right) \sim \text{constant}$$

$\rho T_\perp = P_\perp$, therefore,

$$\Rightarrow \left(\frac{P_\perp}{B\rho}\right) = \text{constant}$$

And in invariant form...

$$\frac{d}{dt}\left(\frac{P_\perp}{B\rho}\right) = 0$$

↑ If you can calculate along the fluid trajectory what B and ρ are, you can determine the transverse pressure

NOW: Need an expression for $P_{\parallel}$

$J_{\parallel} \sim \int dL\, v_{\parallel}$

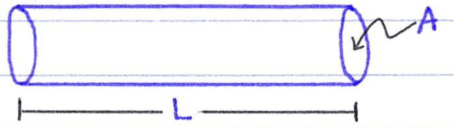
$J_{\parallel}^2 \sim (\int dL\, v_{\parallel})^2$ gives another adiabatic invariant

we will need to do something with this

① $\underbrace{\langle v_{\parallel}^2\rangle}_{\sim T_{\parallel}} L^2 \sim$ constant

$\sim T_{\parallel}$, from which we can find $P_{\parallel}$ as we did above with $T_\perp$ & $P_\perp$

-Dealing with L: Define a flux tube!



└ "Frozen in condition" – can define a flux tube with a plasma on it with a local magnetic field, B

Two quantities must be conserved as the tube moves around:

② Magnetic flux conservation

BA ∼ constant

③ Constant # of particles

$\rho AL \sim$ constant

└ #particles/unit volume.

$\rightarrow L \sim 1/\rho A$

→

Rewrite ①: Use ③ to get rid of L

$$\langle v_\parallel^2\rangle L^2 \sim \langle v_\parallel^2\rangle \left(\frac{1}{\rho A}\right)^2 \underbrace{\left(\frac{B^2}{B^2}\right)}_{\text{insert}} \sim \text{constant}$$

$$\langle v_\parallel^2\rangle \frac{B^2}{\rho^2 \boxed{A^2B^2}} \sim \text{constant}$$

from ② we know this is also a constant

→ absorb into RHS

$$\rightarrow \frac{\langle v_\parallel^2\rangle B^2}{\rho^2} = \text{constant}$$

Eliminate B: Use μ invariance

$$\frac{P_\perp}{B\rho} = \text{constant} \rightarrow B \sim \frac{P_\perp}{\rho}$$

$\langle v_\parallel^2\rangle \sim T_\parallel$ as before

$$\frac{T_\parallel}{\rho^2}\left(\frac{P_\perp}{\rho}\right)^2 \underbrace{\left(\frac{\rho}{\rho}\right)}_{\text{insert}} \sim \text{constant}$$

$\rho T_\parallel = P_\parallel$, therefore

$$\Rightarrow \left(\frac{P_\parallel P_\perp^2}{\rho^5}\right) = \text{constant}$$

→Equations of State for $P_\perp$ and $P_\parallel$←

$$\frac{d}{dt}\left(\frac{P_\perp}{B\rho}\right) = 0 \quad , \quad \frac{d}{dt}\left(\frac{P_\parallel P_\perp^2}{\rho^5}\right) = 0$$

These equations replace $\frac{d}{dt}P + \Gamma P \nabla\cdot\vec{u} = 0$ from the MHD equations.

*Note: We've tossed everything to do with parallel thermal conduction in this problem that might describe how $P_\parallel$ evolves in time
(must assume bounce motion on appropriate timescales because particles cannot just stream along)

→

Putting it all together...

The CGL Equations

- The momentum equation
  - pressure tensor updated to handle parallel anisotropies

$$\rho \frac{d}{dt}\vec{u} = -\nabla\cdot\left[P_{\parallel}\hat{b}\hat{b} + P_{\perp}\left(\overset{\equiv}{I}-\hat{b}\hat{b}\right)\right] - \nabla\left(\frac{B^2}{8\pi}\right) + \frac{1}{4\pi}\vec{B}\cdot\nabla\vec{B}$$

- The continuity equation
  - unchanged

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{u} = 0$$

- Faraday's Law
  - unchanged

$$\frac{\partial\vec{B}}{\partial t} - \nabla\times(\vec{u}\times\vec{B}) = 0$$

- and our two new equations
  - from μ invariance ⤵
  - from $J_{\parallel}$ invariance ⤵

$$\frac{d}{dt}\left(\frac{P_{\perp}}{B\rho}\right) = 0 \qquad \frac{d}{dt}\left(\frac{P_{\parallel}P_{\perp}^2}{\rho^5}\right) = 0$$

*must be careful, especially with electrons, because of lack of thermal conduction in these equations
(i.e., parallel transport is neglected)

These equations are useful when anisotropies are very impactful, such as with the firehose instability.

Firehose Instability: Basic Physics

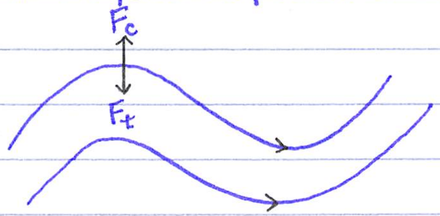
*Important in magnetic reconnection, which drives particles to increase $P_{\parallel}$ but not $P_{\perp}$

Initially:

$P_{\parallel} > P_{\perp}$ e.g., a thin or elongated galaxy

The MHD shear Alfvèn waves in the plasma may become unstable for $\parallel > \perp$

→ this system spontaneously buckles from the instability



Particles streaming along bent field lines feel an effective centrifugal force and a tension force.

- For $F_t > F_c$:

  The magnetic field returns to being straight

- For $F_c > F_t$:

  Firehose instability continues

How do we stabilize such a system?

Stability Conditions:

$$F_c \sim n_0 \frac{m\langle v_{\parallel}^2\rangle}{R} \sim n_0 m \langle v_{\parallel}^2\rangle K$$

radius of curvature of the bent field line

K = curvature vector; $K \sim 1/R$

$$F_t \sim \frac{B^2}{4\pi} K$$

This system is unstable for...

$$F_c > F_t \longrightarrow \underbrace{n_0 m\langle v_{\parallel}^2\rangle}_{T_{\parallel}} > \frac{B^2}{4\pi}$$

So writing this in terms of $P_{\parallel}$, our source of anisotropy

$P_{\parallel} \sim n_0 T_{\parallel}$

$$\frac{P_{\parallel} 4\pi}{B^2} > 1 \Rightarrow \underbrace{\frac{\beta_{\parallel}}{2} > 1}$$

where $\beta = \frac{8\pi P}{B^2}$ from Lec. #21

If this is true, the system will remain unstable

*FOR HW: Should get final answer

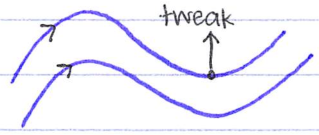
$$\frac{\beta_{\parallel}}{2} - \frac{\beta_{\perp}}{2} > 1$$

from linear analysis

KLE

Suppose the system is at the Firehose threshold, $\beta_{\parallel}/2 \sim 1$

↳ take a field line and tweak it up



⇒ @ threshold, the net restoring force is ZERO!

└ have "floppy" field lines instead of "elastic band" field lines

*this is important because that tension/restoring force is critical in magnetic reconnection.

That was the fluid approach... what about kinetic theory?

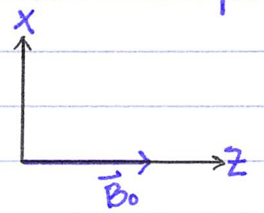
Firehose Instability from Kinetic Theory

→ Previously done in a system lacking an ambient $\vec{B}$-field

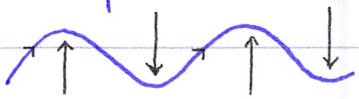
Initially $\vec{B}_0 = \underbrace{B_0\hat{z}}_{\text{straight field lines}}$

with $f_0(v_\perp, v_\parallel)$; no azimuthal dependence

→ Allow the system to have pressure anisotropies



With $\vec{B}$-field perturbations in $\hat{x}$-direction



Must linearize:

$$f_1 = \text{Re}\{\hat{f}e^{ikz - i\omega t}\}$$
↑ 1-D $\vec{k}$-vector in $\hat{z}$ direction

$$\vec{B}_1 = (B_1, 0, 0)$$
B_1: └ perturbation in x only

third component 0: └ no perturbation in z because $\vec{k}\cdot\vec{B}$ must = 0

$$\vec{E}_1 = (0, E_1, 0)$$
└ from Faraday's Law ↴

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{B}_1 + \nabla\times\vec{E}_1 = 0$$

$\frac{\partial}{\partial t} \to -i\omega$ 　 $\nabla \to ik\hat{z}$

$$-\frac{\cancel{i}\omega}{c}B_1\hat{x}+\cancel{i}k\hat{z}\times\hat{y}E_1=0$$

$$+\frac{\omega}{c}B_1\underset{\uparrow}{+}kE_1=0 \Rightarrow \frac{-\omega}{kc}B_1=E_1$$

sign from $\hat{z}\times\hat{y}$

NOW: The Boltzmann Equation

"Not the easy part"

Use f_1 to write the linearized Boltzmann equation

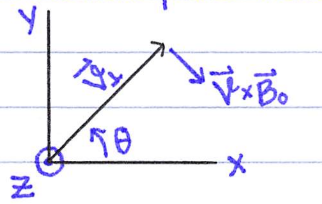
$$(-i\omega+ikv_{\parallel})\hat{f}+\overbrace{\frac{q}{m}\left(\underset{\uparrow}{\hat{E}}\hat{y}+\frac{1}{c}\vec{v}\times\hat{B}\hat{x}\right)\cdot\frac{\partial}{\partial\vec{v}}f_0}^{\text{one of two force terms}}$$

perturbed $\vec{E}$ & $\vec{B}$ dot into f_0

$$+\frac{q}{m}\underset{\uparrow}{\frac{1}{c}}\vec{v}\times\hat{z}\underbrace{B_0\cdot\frac{\partial}{\partial\vec{v}}\hat{f}}_{\text{equilibrium } B_0 \text{ with perturbed } f}=0$$

no equilibrium E_0

– In some polar coordinate system:



So $\vec{v}\times\hat{z}B_0$ is in the $-\hat{\theta}$ direction

Inspecting the unperturbed force term...

$$\frac{q}{m}\frac{1}{c}\underbrace{\vec{v}\times\hat{z}B_0\cdot\frac{\partial}{\partial\vec{v}}\hat{f}}_{=-B_0v_\perp\frac{1}{v_\perp}\frac{\partial}{\partial\theta}\hat{f}}=-\underset{\uparrow}{\Omega}\frac{\partial}{\partial\theta}\hat{f}$$

Cyclotron frequency $= qB/mc$

where $\partial/\partial\theta$ is a velocity-space derivative, not position-space

What about the perturbed force?

$$\vec{v}\times\hat{B}\hat{x}\cdot\partial/\partial\vec{v}\,f_0\rightarrow\vec{v}\times\hat{x}\cdot\underbrace{\partial/\partial\vec{v}\,f_0}$$

Want to change this velocity derivative so that it is acting on $v_{\parallel}^2$ and $v_{\perp}^2$ (the arguments of $f_0 = f_0(v_\parallel^2, v_\perp^2)$ where $v_\perp^2 = v_x^2 + v_y^2$)

↷ $\hat{y}$ does not act on $v_\parallel$

$$\vec{v}\times\hat{x}\cdot\frac{\partial}{\partial\vec{v}}f_0 = v_\parallel\underbrace{\hat{y}\cdot\left(\frac{\partial}{\partial\vec{v}}v_\perp^2\right)}\frac{\partial f_0}{\partial v_\perp^2} - v_y\hat{z}\cdot\underbrace{\left(\frac{\partial}{\partial\vec{v}}v_\parallel^2\right)}_{2v_\parallel}\frac{\partial f_0}{\partial v_\parallel^2}$$

this term is a y-deriv.

$2v_y$; where $v_y = v_\perp \sin(\theta)$

$$= 2v_\parallel v_\perp \sin(\theta)\underbrace{\left(\frac{\partial f_0}{\partial v_\perp^2} - \frac{\partial f_0}{\partial v_\parallel^2}\right)}$$

⇒ This is the source of $P_\parallel$ & $P_\perp$

(in an isotropic plasma this would cancel)

* We can use this same approach for $\hat{E}\hat{y}$

RECAP: What we have so far...

$$\left(-i\omega + ikv_\parallel - \underbrace{\Omega_0\frac{\partial}{\partial\theta}}\right)\hat{f} + \frac{q}{m}\left[\overbrace{2\hat{E}v_\perp\sin(\theta)\frac{\partial f_0}{\partial v_\perp^2}}^{\text{from } \hat{E}\hat{y}} + 2\hat{B}v_\parallel v_\perp\left(\frac{\partial f_0}{\partial v_\perp^2} - \frac{\partial f_0}{\partial v_\parallel^2}\right)\right] = 0$$

treatment of this new term is the main thing here

- We can get rid of $\hat{B}$ using $\hat{B} = (-kc/\omega)\hat{E}$
  - ↳ pull out $\hat{E}$, leaving the RHS in terms of $\sin(\theta)$

Next time → Use Ampère's Law to find the current