Lecture 22 – Gravitational Instability

11/09/17

Guest Lecturer: Dr. Mark Swisdak

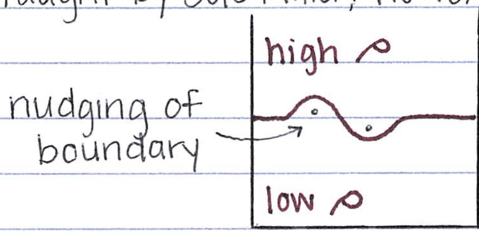
Rayleigh-Taylor Instability

– a.k.a. gravitational instability

Rayleigh-Taylor (RT) is the instability of the interface between two fluids of different densities superimposed one over the other

From ASTR680 – High Energy Astrophysics:

{Taught by Cole Miller, no textbook}



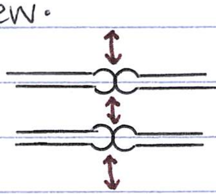
Different densities superimposed have instabilities along their interface

Magnetically, this means it is energetically favorable for matter to nudge the lines around than to try to break through the lines.

· side view ·

It's hard to push here – that is, it's very difficult for matter to break through lines

· top view ·



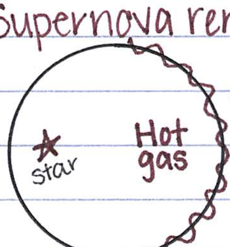
It's much easier to push here – that is, it takes much less energy to nudge these lines around

(returning to PHYS761...)

Examples:

– Mushroom cloud

– Supernova remnants



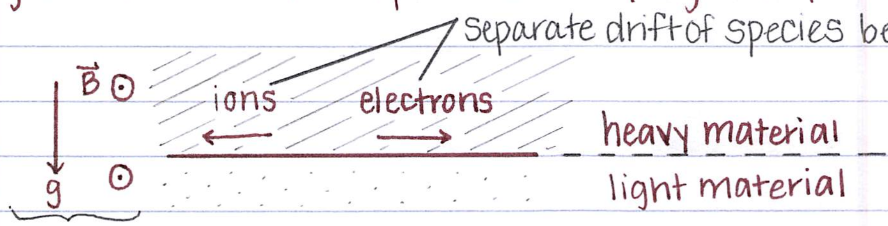
dense ISM

observed R-T instability @ density gradient interface

KLE

KLE

Rosenbluth & Longmire (1957) showed a plasma in an [effective] gravitational field experiences Rayleigh-Taylor instability.



*Note: R-T instability only when heavy is above light w.r.t. $\hat{g}$

plasma layer can be supported against gravity by a $\vec{B}$-field (dependent on $\vec{B}$-field orientation)

Electron and Ion drift:

$$\vec{V} = \frac{c}{qB^2} \overbrace{\vec{F}}^{\text{gravity}} \times \vec{B}$$

charge determines direction of particle drift

*Similar to $\vec{V}_{\perp 0}$ from Lec. #17;
$\vec{B} \times (q/c\, \underbrace{\vec{V}_\perp} \times \vec{B} = -\vec{F}_\perp)$

$\vec{k}$ along $\hat{b}$ does not affect the development of R-T instability

- this balances out to lowest order

$$0 = \rho\vec{g} + \frac{1}{c}(\vec{J} \times \vec{B})$$
$$= n\,(\text{Net Force})$$

$\vec{B}$-field plays a larger role in supporting plasma against gravity in regions of higher density

$$\Rightarrow \vec{J} = c\rho \frac{(\vec{g} \times \vec{B})}{B^2}$$

Orientation of $\vec{B}$ with respect to $\vec{g}$ determines current within the plasma

heavy
light ⊙$\vec{B}$ Same result as no $\vec{B}$-field

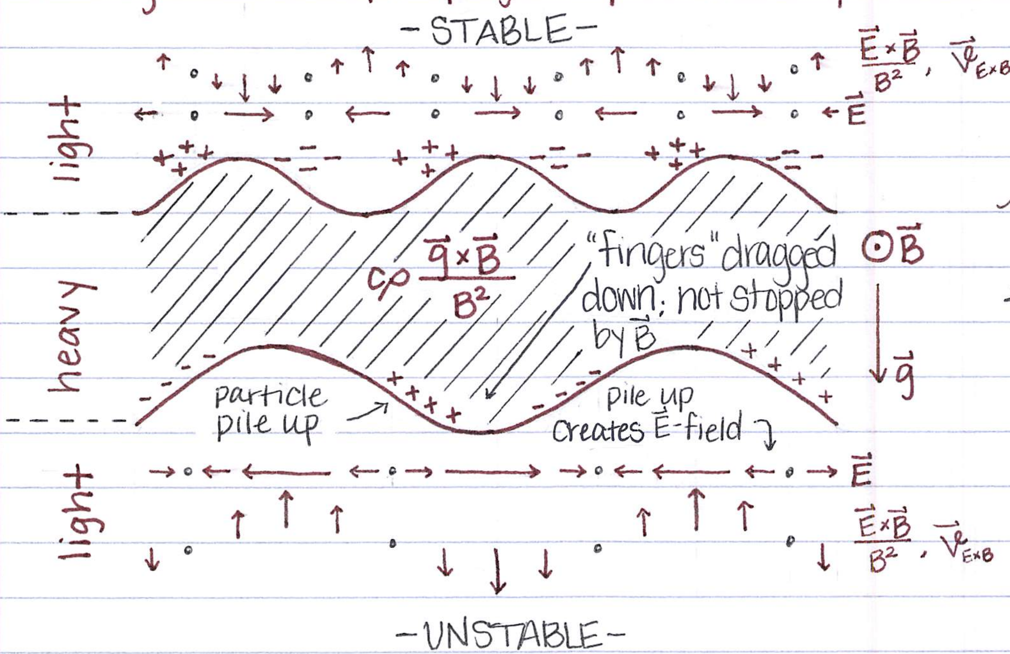
$\vec{k}$ chosen to be $\perp \vec{B}$ here

heavy
light $\rightarrow \vec{B}$ Instability still can occur

a different $\vec{k}$ must be chosen in order not to perturb $\vec{B}$

*Note: A rotating / twisting $\vec{B}$-field is the only arrangement that allows stabilization

We can visualize the geometry for a simple physical description of the gravitational / Rayleigh-Taylor instability:

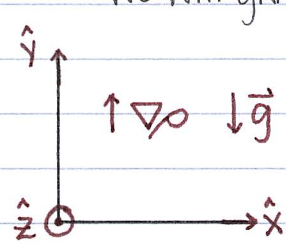


$k_z = 0$ implied (wave # of the disturbance

Equilibria

To analyze stability, we must start with an equilibrium and then linearize our equations around that equilibrium

↳ we will give several examples



0th order, no perturbation

$\vec{B} = B_0(y)\hat{z}$

$\rho = \rho_0(y)$

$P = P_0(y)$; where $\frac{dP}{dy} = -\rho g$ in hydrostatic equilibrium in the absence of $\vec{B}$

Force Balance

To 0th order — momentum conservation in the vertical direction ($\hat{y}$)

$$0 = \frac{-\partial}{\partial y}\left(P_0 + \frac{B_0^2}{8\pi}\right) - \rho_0 g$$

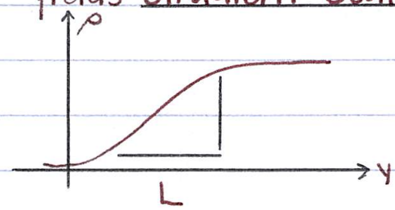
where the full momentum conservation equation here is

$$\rho\underbrace{\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right)}^{\vec{u}=0 \text{ for ideal equilibria (no flow)}} = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B} - \rho g$$

where $(\vec{B}\cdot\nabla)\vec{B} = B^2\vec{\kappa}$ (from Lec. #19), but $\vec{\kappa} = 0$ for our straight field lines.

How does our density change with height?

↳ yields <u>Gradient Scale Length</u>



$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial y} \sim \frac{1}{L}$$

such that we may make the scaling/dimensional argument

$$\nabla \sim 1/L$$

• System timescales:

– from the quantities L, g, we have 3 system timescales

$$\gamma_{\substack{\text{growth}\\\text{rate}}} = \gamma_g \sim \underbrace{\sqrt{\frac{g}{L}}}_{\text{this is just } \omega \text{ for a simple pendulum!}}$$

$$\gamma_{\substack{\text{sound}\\\text{waves}}} = \gamma_s \sim \frac{C_s}{L}$$

fast-mode propagation time

$$\gamma_{\substack{\text{fast}\\\text{mode}}} = \gamma_f \sim \frac{C_F}{L}, \quad C_F^2 = C_A^2 + C_S^2$$

where the fast mode is a combination of both magnetic and plasma pressure

** If possible, this mode should be used primarily over γ_s

Assume $\gamma_g \ll \gamma_s \,/\, \gamma_g \ll \gamma_F$ (weak gravitation)

⇒ This is the Bouissenesq Approximation

• {from Wiki} In the field of buoyancy-driven flow/natural convection, it ignores density differences except where they appear in terms multiplied by $\vec{g}$. The essence of the Bouissenesq approximation is that the differences in inertia between two fluids is negligible but gravity is sufficiently strong to make the specific

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weights appreciably different → Sound waves are impossible/neglected when the Bouissenesq approximation is used since sound waves move via density variations

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\vec{u}) = 0$$

└ continuity eq. for conservation of mass

If density variations are ignored via the Bouissenesq approximation, this reduces to:

$$\nabla\cdot\vec{u} \approx 0 \Rightarrow \text{nearly incompressible motion!}$$

The MHD Equations: (a recap for moving forward)

① Conservation of momentum

$$\rho\frac{\partial\vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla\underbrace{\left(P + \frac{B^2}{8\pi}\right)}_{} + \frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B} - \rho\vec{g}$$

This term is very large compared to the others and needs to be annihilated

② Faraday's Law

$$\frac{\partial\vec{B}}{\partial t} - \nabla\times(\vec{u}\times\vec{B}) = 0$$

($\vec{E}$-field eqn. already substituted in)

③ Conservation of energy (the pressure/energy eqn.)

$$\frac{\partial P}{\partial t} + (\vec{u}\cdot\nabla)P + \Gamma P(\nabla\cdot\vec{u}) = 0$$

└ Ratio of specific heats

$\Gamma = (n_f + 2)/n_f$ — #D.O.F.

④ Conservation of mass

$$\frac{\partial\rho}{\partial t} + (\vec{u}\cdot\nabla)\rho + \rho(\nabla\cdot\vec{u}) = 0$$

Linearized Equations

The above equations are what we want to linearize about our equilibrium.

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$\rho = \rho_0 + \rho_1$

complex amplitude

$\rho_1(x,y,t) = \mathrm{Re}\{\hat{\rho}(y)\, e^{\underbrace{ikx - i\omega t}}\}$

only variation in x; $\partial/\partial z = 0$

$\vec{B} = \vec{B}_0 + \vec{B}_1$

assumed $\vec{B}_1 = \mathrm{Re}\{\hat{\vec{B}}_1(y)\, e^{ikx - i\omega t}\}$

$\vec{k} = k\hat{i}$

from this, $\vec{B}_1 = B_1(x,y)\hat{z}$

$\underbrace{(\vec{B}_1 \cdot \nabla)}_{\hat{z}}\overbrace{\vec{B}_0}^{\hat{z}}$, where $\partial/\partial z = 0$, $= 0$

- Equation ①

$\rho \frac{\partial}{\partial t}\vec{u} + \rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla\left(P + \frac{1}{8\pi}B^2\right) + \frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B} - \rho\vec{g}$

Where $(\vec{u}\cdot\nabla)\vec{u} = u_{1y}\frac{\partial}{\partial y}\vec{u} = 0$

linearizing... P_0 goes away with $\rho_0 g$; the equilibrium term from Force Balance

Perturbing $B^2 \to \vec{B}_0^2 + 2\vec{B}_0\cdot\vec{B}_1 + \vec{B}_1^2$

$$\underbrace{\rho_0\frac{\partial \vec{u}_1}{\partial t}} = -\left(P_1 + \frac{\vec{B}_0\cdot\vec{B}_1}{4\pi}\right) + \frac{1}{4\pi}\left[\cancelto{0}{(\vec{B}_1\cdot\nabla)\vec{B}_0} + \cancelto{0}{(\vec{B}_0\cdot\nabla)\vec{B}_1}\right] - \rho_1\vec{g}$$

Perturbation is 1st order only; equilibrium terms like $\vec{u}_0$ are gone

$$\rho_0\frac{\partial \vec{u}_1}{\partial t} = -\nabla\left(P_1 + \frac{\vec{B}_0\cdot\vec{B}_1}{4\pi}\right) - \rho_1\vec{g}$$

- Equation ②

$\frac{\partial}{\partial t}\vec{B} - \nabla\times(\vec{u}\times\vec{B})$

$$\frac{\partial\vec{B}_1}{\partial t} - \nabla\times(\vec{u}_1\times\vec{B}_0) = 0 = \frac{\partial\vec{B}_1}{\partial t} - \underbrace{\cancelto{0}{(\vec{B}_1\cdot\nabla)\vec{u}_1}} + \overbrace{(\vec{u}_1\cdot\nabla)\vec{B}_0}^{=u_{1y}\frac{\partial B_0}{\partial y}} + \vec{B}_0(\nabla\cdot\vec{u}_1) - \vec{u}_1\cancelto{0}{(\nabla\cdot\vec{B}_0)}$$

$\frac{\partial}{\partial t}\vec{B}_0 = 0$

$\vec{B}_1\cdot\nabla$ must $= 0$ for $(\vec{B}_1\cdot\nabla)\vec{B}_0 = 0$ to hold true for all values of $\vec{B}_0$

$$\frac{\partial\vec{B}_1}{\partial t} + \vec{B}_0(\nabla\cdot\vec{u}_1) + u_{1y}\frac{\partial}{\partial y}\vec{B}_0 = 0$$

$\rightarrow$

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- Equation ③

$$\frac{\partial}{\partial t}P + (\vec{u}\cdot\nabla)P + \Gamma P(\nabla\cdot\vec{u}) = 0$$

$$\frac{\partial P_1}{\partial t} + \overbrace{(\vec{u}_1\cdot\nabla)P_0}^{= u_{1y}\,\partial/\partial y\, P_0} + \Gamma P_0(\nabla\cdot\vec{u}_1) = 0$$

$\frac{\partial}{\partial t}P_0 = 0$

$$\frac{\partial P_1}{\partial t} + u_{1y}\frac{\partial P_0}{\partial y} + \Gamma P_0(\nabla\cdot\vec{u}_1) = 0$$

- Equation ④

$$\frac{\partial}{\partial t}\rho + (\vec{u}\cdot\nabla)\rho + \rho(\nabla\cdot\vec{u}) = 0$$

$$\frac{\partial \rho_1}{\partial t} + \overbrace{(\vec{u}_1\cdot\nabla)\rho_0}^{= u_{1y}\,\partial/\partial y\,\rho_0} + \rho_0(\nabla\cdot\vec{u}_1) = 0$$

$\frac{\partial}{\partial t}\rho_0 = 0$

$$\frac{\partial \rho_1}{\partial t} + u_{1y}\frac{\partial \rho_0}{\partial y} + \rho_0\underbrace{(\nabla\cdot\vec{u}_1)} = 0$$

We need to figure out how big this is
(want to show the $\nabla\cdot\vec{u}\approx 0$ approximation works here)

⇒ Use low frequency ordering

- from equation ③

say these two terms are comparable

$$\underbrace{\frac{\partial P_1}{\partial t}} + \underbrace{u_{1y}\frac{\partial P_0}{\partial y}} + \cancel{\Gamma P_0(\nabla\cdot\vec{u}_1)}^{\to 0} = 0$$

$\partial/\partial y \to \frac{1}{L}$, the length scale of the instability

$\partial/\partial t \to \gamma_g$, the timescale of the instability

$$\gamma_g P_1 \sim u_1\frac{P_0}{L}$$

Supports that $P_1 + \frac{B_0\cdot B_1}{4\pi}$ is large.

→ then, comparing the left-hand-side & P_1 term in equation ①

$\nabla \sim 1/L$ as previously stated

$$\rho_0\frac{\partial \vec{u}_1}{\partial t} = -\nabla\left(P_1 + \underbrace{\frac{\vec{B}_0\cdot\vec{B}_1}{4\pi}}\right) - \cancel{\rho_1 g}$$

small

same order as P_1

$\rho_0 \gamma_g u_1 \sim \frac{1}{L} \left(P_1 \left(\frac{\gamma_g}{\gamma_g} \right) \right)$ — $\gamma_g P_1 = u_1 P_0 / L$

$\rho_0 \gamma_g \cancel{u_1} \sim \frac{1}{L} \frac{\cancel{u_1}}{L} \frac{P_0}{\gamma_g}$

$\gamma_g^2 \sim \frac{P_0}{\rho_0 (L^2)} \sim \gamma_s$

$\gamma_g^2 \sim \gamma_s^2 \rightarrow$ implies that $P_1 + \frac{B_0 \cdot B_1}{4\pi}$ is huge

To lowest order

$\hookrightarrow P_1 + \frac{B_0 B_1}{4\pi} \simeq 0$

The pressure from the plasma & the $\vec{B}$-field must balance / very nearly cancel out because the other terms are small

NOW: Add $(B_0/4\pi)$ ② + ③

$\rightarrow$ gives equations for B_1 & P_1; yields above combination

$$\left(\frac{\vec{B}_0}{4\pi}\right)\left[\frac{\partial \vec{B}_1}{\partial t} + \vec{B}_0(\nabla\cdot\vec{u}_1) + u_{1y}\frac{\partial \vec{B}_0}{\partial y}\right] + \left[\frac{\partial P_1}{\partial t} + u_{1y}\frac{\partial P_0}{\partial y} + \Gamma P_0(\nabla\cdot\vec{u}_1)\right]$$

$$= \frac{\partial}{\partial t}\left(P_1 + \frac{\vec{B}_0\vec{B}_1}{4\pi}\right) + u_{1y}\overbrace{\underbrace{\left[\frac{\partial P_0}{\partial y} + \frac{\partial}{\partial y}\left(\frac{B^2}{8\pi}\right)\right]}_{= \partial/\partial y (P_0 + B_0^2/8\pi) = -\rho_0 \vec{g}}}^{\text{use pressure balance}} + (\nabla\cdot\vec{u}_1)\left[\frac{B_0^2}{4\pi} + \Gamma P_0\right] = 0$$

the 0^{th} order momentum conservation eq.

$\downarrow$ Solve for $\nabla\cdot\vec{u}_1$ $\downarrow$

$$(\nabla\cdot\vec{u}_1)\left[\frac{B_0^2}{4\pi} + \Gamma P_0\right] = u_{1y}\rho_0 g - \frac{\partial}{\partial t}\left(P_1 + \frac{\vec{B}_0\cdot\vec{B}_1}{4\pi}\right)$$

$$\nabla\cdot\vec{u}_1 = \frac{u_{1y}\rho_0\vec{g} - \cancel{\frac{\partial}{\partial t}\left(P_1 + \frac{\vec{B}_0\cdot\vec{B}_1}{4\pi}\right)}^{\text{small}}}{\left[\frac{B_0^2}{4\pi} + \Gamma P_0\right]}$$

As stated above, these large terms must approximately cancel each other out (to first order)

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$$\nabla\cdot\vec{u}_1 = u_{1y}\vec{g}\frac{\rho_0}{(B_0^2/4\pi+\Gamma P_0)} = \frac{u_{1y}\vec{g}}{C_A^2+C_S^2} \sim \frac{u_{1y}}{L}\frac{\gamma_g^2}{\gamma_f^2}$$

$(C_A^2+C_S^2)=C_f^2$

$C_S^2=\frac{\Gamma P_0}{\rho_0}$

$C_A^2=\frac{B_0^2}{4\pi\rho_0}$

for $\gamma_g \ll \gamma_f$

$$\frac{u_{1y}}{L}\frac{\gamma_g^2}{\gamma_f^2} \ll \frac{u_{1y}}{L}$$

$\nabla\cdot\vec{u}_1 \ll \frac{u_{1y}}{L}$ ⇒ Nearly incompressible!

↳ From this we know it is valid under these conditions to use the Bouissenesq appx., $\nabla\cdot\vec{u}\simeq 0$

We must annihilate the pressure term in Eq. ①

↳ Operate with $\hat{z}\cdot\nabla\times$①

$$\hat{z}\cdot\nabla\times\left[\rho_0\frac{\partial\vec{u}_1}{\partial t}\right] = \hat{z}\cdot\nabla\times\Big[\underbrace{\cancel{-\nabla\left(P_1+\frac{\vec{B}_0\cdot\vec{B}_1}{4\pi}\right)}^{\,0}}_{\nabla^2\left(P_1+\frac{\vec{B}_0\cdot\vec{B}_1}{4\pi}\right)=0}-\rho_1\vec{g}\Big]$$

– taking the curl is the only way to guarantee annilation of this large pressure term

$$\hat{z}\cdot\nabla\times\left(\rho_0\frac{\partial}{\partial t}\vec{u}_1\right) = -\hat{z}\cdot\nabla\times(\rho_1 g\hat{y})$$

$$= -g\frac{\partial\rho_1}{\partial x}$$

$$\hat{z}\cdot\nabla\times\left(\rho_0\frac{\partial\vec{u}_1}{\partial t}\right) = \underbrace{\frac{\partial}{\partial x}}_{\partial/\partial x\to ik}\left(\rho_0\frac{\partial u_{1y}}{\partial t}\right) - \frac{\partial}{\partial y}\left(\rho_0\frac{\vec{u}_{1x}}{\underbrace{\partial t}_{\partial/\partial t\to -i\omega}}\right) = -g\frac{\partial\rho_1}{\partial x}$$

$$ik(-i\omega)\rho_0\hat{u}_{1y} + i\omega\frac{\partial}{\partial y}(\rho_0\hat{u}_{1x}) = -gik\hat{\rho}_1$$

– use eq. ④ with $\nabla\cdot\vec{u}=0$

$$\frac{\partial}{\partial t}\rho_1 + \hat{u}_{1y}\frac{\partial}{\partial y}\rho_0 + \rho_0\cancel{(\nabla\cdot\vec{u})}^{\,0} = 0$$

$$i\omega\rho_1 = \hat{u}_{1y}\frac{\partial}{\partial y}\rho_0$$

↓plugging into our expression from eq. ①↓

$$k\omega\rho_0\hat{u}_{1y} + i\omega\frac{\partial}{\partial y}(\rho_0\hat{u}_{1x}) = -g\cancel{i}k\frac{\hat{u}_{1y}\frac{\partial}{\partial y}\rho_0}{\cancel{i}\omega}$$

$$\omega^2\Big[k\rho_0\hat{u}_{1y} + i\frac{\partial}{\partial y}(\underbrace{\rho_0\hat{u}_{1x}}_{\text{want to get everything in y-components}})\Big] = -gk\hat{u}_{1y}\frac{\partial}{\partial y}\rho_0$$

→Linearize eq. ④

$$ik\hat{u}_{1x} + \frac{\partial}{\partial y}(\hat{u}_{1y}) = 0 \Rightarrow \hat{u}_{1x} = \frac{-1}{ik}\frac{\partial}{\partial y}(\hat{u}_{1y})$$

↓substituting for $\hat{u}_{1x}$↓

$$\omega^2\left[k\rho_0\hat{u}_{1y} + \cancel{i}\frac{\partial}{\partial y}\left(\rho_0\left(\frac{-1}{\cancel{i}k}\frac{\partial}{\partial y}\hat{u}_{1y}\right)\right)\right] = -gk\hat{u}_{1y}\frac{\partial}{\partial y}\rho_0$$

multiply through by $-k$

$$\omega^2\left[\frac{\partial}{\partial y}\left(\rho_0\frac{\partial}{\partial y}\hat{u}_{1y}\right) - k^2\rho_0\hat{u}_{1y}\right] = gk^2\hat{u}_{1y}\frac{\partial\rho_0}{\partial y}$$

⇒ No dependence on the $\vec{B}$-field at all!

supports our statement that $\odot\vec{B}$ w.r.t. $\downarrow\vec{g}$ has the same effects as no field at all

Limiting Solutions

We want to solve equation ⑤ under two limits:

- First take $\underbrace{kL \gg 1}_{\text{this is the short wavelength perturbation}}$

$1/L \sim \partial/\partial y$ so $\partial/\partial y \ll k$

Eqn. ⑤ then becomes

$$\omega^2\Big[\underbrace{\frac{\partial}{\partial y}\left(\rho_0\frac{\partial}{\partial y}\hat{u}_{1y}\right)} - \underbrace{k^2\rho_0\hat{u}_{1y}}\Big] = gk^2\hat{u}_{1y}\frac{\partial\rho_0}{\partial y}$$

This term, with two factors of $\partial/\partial y$, is much SMALLER than

$$\rightarrow -\omega^2 \cancel{k^2} \rho_0 \cancel{\hat{u}_{1y}} = g \cancel{k^2} \cancel{\hat{u}_{1y}} \frac{\partial \rho_0}{\partial y}$$

$$\omega^2 = -\boxed{\frac{g}{\rho_0} \frac{\partial \rho_0}{\partial y}}$$

~ $1/L$, the gradient scale length

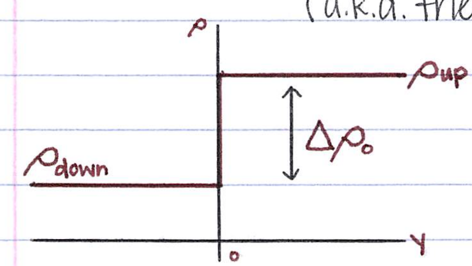
$= -g/L = -\gamma_g^2$

unstable if $\partial \rho_0 / \partial y > 0$
(heavy fluid above light)

⇒ Heavy fluid falling in a gravitational field releases energy

$\gamma^2 =$ timescale of the instability $= \gamma_g^2$

• Second, take $\underbrace{kL \ll 1}$
this is the long wavelength perturbation
(a.k.a. the sharp boundary limit)



– Away from the boundary (jump)

$\rho_0(y) = \rho_0 \rightarrow \partial \rho_0 / \partial y = 0$

Eqn. ⑤ then becomes

$$\omega^2 \left[\frac{\partial}{\partial y}\left(\rho_0 \frac{\partial}{\partial y} \hat{u}_{1y}\right) - k^2 \rho_0 \hat{u}_{1y} \right] = g k^2 \hat{u}_{1y} \cancel{\frac{\partial \rho_0}{\partial y}}^{\,0} = 0$$

pull out

$$\omega^2 \rho_0 \left[\frac{\partial^2}{\partial y^2} \hat{u}_{1y} - k^2 \hat{u}_{1y} \right] = 0$$

$$\cancel{\omega^2 \rho_0} \left[\frac{\partial^2}{\partial y^2} - k^2 \right] \hat{u}_{1y} = 0$$

$$\rightarrow \hat{u}_{1y} = \begin{cases} u_{y_0} e^{-iky}, & y > 0 \\ u_{y_0} e^{iky}, & y < 0 \end{cases}$$

→

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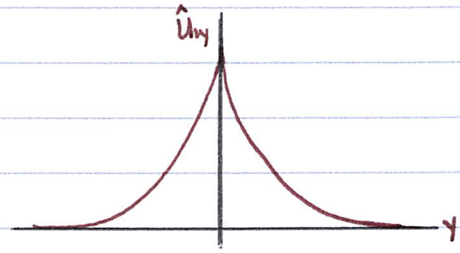


*the slope of $\hat{u}_y$ undergoes a jump but the magnitude of $\hat{u}_y$ does not

– Near the boundary (jump)

$\partial/\partial y \gg k^2$

Eqn. 5 then becomes

$$\omega^2\Big[\underbrace{\frac{\partial}{\partial y}\Big(\rho_0 \frac{\partial}{\partial y}\hat{u}_{1y}\Big)}\; - \underbrace{k^2 \rho_0 \hat{u}_{1y}}\Big] = g k^2 \hat{u}_{1y}\frac{\partial \rho_0}{\partial y}$$

This term, with two factors of $\partial/\partial y$, is much LARGER than

$$\rightarrow \omega^2\Big[\frac{\partial}{\partial y}\rho_0 \frac{\partial}{\partial y}\hat{u}_{1y}\Big] = g k^2 \frac{\partial \rho_0}{\partial y}\hat{u}_{1y}$$

Must integrate over the boundary $\int_{-\epsilon}^{\epsilon} dy$

$$\omega^2 \int_{-\epsilon}^{\epsilon}\Big[\frac{\partial}{\partial y}\rho_0 \frac{\partial}{\partial y}\hat{u}_{1y}\Big]dy = \omega^2 \rho_0 \underbrace{\frac{\partial}{\partial y}\hat{u}_{1y}}\Big|_{-\epsilon}^{\epsilon}$$

use "away from boundary" solution

$$= \omega^2\big[\rho_0^{up}(-k\hat{u}_{1y}) - \rho_0^{down}(k\hat{u}_{1y})\big]_{y=0}$$

$$= -\omega^2 k \hat{u}_{1y}(0)\big[\rho_{up} + \rho_{down}\big]$$

$$\int_{-\epsilon}^{\epsilon} g k^2 \hat{u}_{1y}\frac{\partial \rho_0}{\partial y}dy = g k^2 \hat{u}_{1y}(0)\big[\rho_{up} - \rho_{down}\big]$$

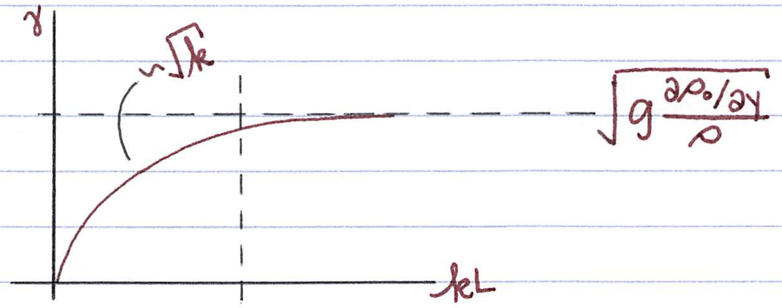
↓ putting this together ↓

$$-\omega^2 \cancel{k}\cancel{\hat{u}_{1y}(0)}\big[\rho_{up} + \rho_{down}\big] = g k^{\cancel{2}} \cancel{\hat{u}_{1y}(0)}\big[\rho_{up} - \rho_{down}\big]$$

$$\omega^2 = -gk \underbrace{\frac{[\rho_{up} - \rho_{down}]}{[\rho_{up} + \rho_{down}]}}$$

If $\rho_{up} > \rho_{down}$, there is an instability

$$\gamma^2 = \frac{gk\Delta\rho_0}{[\rho_{up} + \rho_{down}]}$$



* Notice that the magnetic field is not able to stop the growth of the instability. $\vec{B}_0$ is simply convected with the flow.

In "real life":

└ including small-scale effects ?

→ surface tension

→ viscosity