Lecture 19 - The Magnetohydrodynamics Equations 10/31/17

Happy Halloween!

from last time...

Deriving the MHD Equations

The MHD equations underly work in both lab physics and space & astrophysics

Ordering: p^+/e^- masses comparable

plasma velocity

$u \sim v_t \sim C_A$

Alfvén speed ($v \approx C_A$ for $C_A \ll c$, $v \approx c$ for $C_A \gg c$)

$C_A \sim B/\sqrt{4\pi m n}$

diverges for small n; problematic for relativistic MHD (impacts Ampère's Law)

*Ordering designed to make certain terms small so we can throw them out for the MHD equations

collision rate

$\nu \sim \Omega$, $\omega = v_t/L$ L = macro scale length

$\epsilon = \omega/\nu \sim v_t^2/c^2 \sim r_L/L \ll 1$

small parameter (underlies entire derivation)

⇒ Use this scheme to break down our constituent equations

- The Boltzmann Equation

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla f_\alpha + \left(\frac{q}{m}\right)_\alpha \left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial}{\partial \vec{v}} f_\alpha = \overbrace{C_{\alpha\alpha} + C_{\alpha\beta}}^{\text{collisions}}$$

rank: ϵ ϵ 1 1 1 1

- Gauss' Law

deviation from charge neutrality

$$\nabla\cdot\vec{E} = 4\pi e\left(\int d\vec{v} f_i - \int d\vec{v} f_e\right)$$

rank: ϵ^2 $\underbrace{1 \quad\quad 1}$

the basis for quasineutrality → electron & ion densities ~ equal

⇒ Charge neutral @ 0^{th} & 1^{st} orders

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*Note: Ordering does NOT mean $\vec{E}=0$, just that our scale lengths are much larger than λ_D

- Faraday's Law

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0$$

rank: 1 1

- Ampère's Law

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} e \int d\vec{v}\,\vec{v}\,(f_i - f_e)$$

rank: ϵ ϵ^2 1 1

└ "toss forever" – not in 0^{th} or 1^{st} orders

*doing this does make this not Lorentz invariant, so this is only in the non-relativistic case

Then...

Lowest Order (Rank 1): (ϵ, ϵ^2 thrown out)

① Boltzmann

$$\left(\frac{q}{m}\right)_\alpha \left(\vec{E}_0 + \frac{1}{c}\vec{v}\times\vec{B}_0\right)\cdot\frac{\partial}{\partial \vec{v}} f_{\alpha_0} = \underbrace{C_{\alpha\alpha_0} + C_{\alpha\beta_0}}$$

presence of these terms makes temperatures equal

② Gauss

$$0 = n_i^0 - n_e^0$$

③ Faraday

$$\frac{1}{c}\frac{\partial \vec{B}_0}{\partial t} + \nabla \times \vec{E}_0 = 0$$

④ Ampère

$$0 = (n_i \vec{u}_i)_0 - (n_e \vec{u}_e)_0$$

$$\Rightarrow \vec{J}_0 = 0$$

From ① → Maxwellian, $n_e = n_i$ thus eliminating $C_{\alpha\alpha_0}, C_{\alpha\beta_0}$

$u_{i_0} = u_{e_0} = \vec{u}_0$, $T_e = T_i$

└ f_α is a drifting Maxwellian

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$$\left(\frac{q}{m}\right)_\alpha \left(\vec{E}_0 + \frac{1}{c}\vec{v}\times\vec{B}_0\right)\cdot\frac{\partial}{\partial\vec{v}} f_{\alpha_0} = C_{\alpha\alpha_0} + \cancel{C_{\alpha\beta_0}}^{\,0}$$

$$\hookrightarrow \left(\vec{E}_0 + \frac{1}{c}\vec{v}\times\vec{B}_0\right)\cdot\underbrace{(\vec{v}-\vec{u}_0)}_{} = 0$$

can simplify $\partial/\partial\vec{v} f_{\alpha_0}$ to this b/c f_{α_0} is Maxwellian

$$\vec{E}_0\cdot(\vec{v}-\vec{u}_0) - \frac{1}{c}\vec{u}_0\cdot\vec{v}\times\vec{B}_0 = 0$$

consolidate terms in $\vec{v}$

$$\left(\vec{E}_0 + \frac{1}{c}\vec{u}_0\times\vec{B}_0\right)\cdot\vec{v} = \vec{E}_0\cdot\vec{u}_0$$

This must be valid for any $\vec{v}$ so...

$$\hookrightarrow \underbrace{\vec{E}_0 + \frac{1}{c}\vec{u}_0\times\vec{B}_0}_{} = 0$$

*note: $\vec{E}_0 = 0$ in the plasma frame (the frame moving @ $\vec{v} = \vec{u}_0$; this is the mark of a good conductor)

- $\vec{u}_{\perp 0}$ is the $\vec{E}_0\times\vec{B}_0$ drift

$$\hookrightarrow E_{0\parallel} u_{0\parallel} = 0 \Rightarrow E_{0\parallel} = 0$$

$$\vec{E}_0 = -\frac{1}{c}\vec{u}_0\times\vec{B}_0$$

from $\vec{u}_0$ we can determine $\vec{E}_0$

$\underbrace{n_0, P_0, u_0/E_0}_{\text{the moments of the Boltzmann equation}}$ as yet undetermined

When we go to 1st order we need to deal with collisions. Therefore we will also need...

Symmetry Properties of the Collision Operator

$$\int d\vec{v}\, C_{\alpha\alpha}\begin{pmatrix} 1 \\ m_\alpha\vec{v} \\ \frac{1}{2}m_\alpha v^2 \end{pmatrix} = 0$$

self-collisions (collisions within a species) do not change # density, momentum-, or energy-density

$$\int d\vec{v}\, C_{\alpha\beta}\begin{pmatrix} 1 \\ m_\alpha\vec{v} \\ \frac{1}{2}m_\alpha v^2 \end{pmatrix} = \begin{matrix} 0 \leftarrow \text{\# density unchanged} \\ -\vec{R}_{\alpha\beta} \leftarrow \text{momentum conservation} \\ -(\partial W/\partial t)_{\alpha\beta} \leftarrow \text{total energy unchanged} \end{matrix}$$

$\vec{R}_{\alpha\beta}$ = drag between species α and β

$\hookrightarrow \vec{R}_{\alpha\beta} + \vec{R}_{\beta\alpha} = 0 \Rightarrow$ momentum from $\alpha\to\beta$ = momentum from $\beta\to\alpha$

(see Lecture #6 for definition)

$\rightarrow$

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$$\left(\frac{\partial W}{\partial t}\right)_{\alpha\beta} + \left(\frac{\partial W}{\partial t}\right)_{\beta\alpha} = 0$$ $\Rightarrow$ energy exchange from $\alpha \rightarrow \beta$ = energy exchange from $\beta \rightarrow \alpha$

NOW ↴

First Order in ε

{ Evaluate rank ε terms @ 0^{th} order
{ Evaluate rank 1 terms @ 1^{st} order

To what order do we evaluate f_α? $\rightarrow 0^{th}$

$$\frac{\partial}{\partial t} f_{\alpha_0} + \vec{v}\cdot\nabla f_{\alpha_0} + \left(\frac{q}{m}\right)_\alpha \left[\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial}{\partial\vec{v}} f_\alpha\right]_{1\theta} = \underbrace{C_{\alpha\alpha_1} + C_{\alpha\beta_1}}_{\text{use symmetry to get rid of this}}$$

evaluate @ 1^{st} order

- Integrate over the velocity (operate with $\int d\vec{v}$) and use the fact that $\partial/\partial\vec{v}\cdot\vec{F} = 0$ and that collisions don't create or destroy particles

Properties of the collisions:

$$\int d\vec{v}\, C_{\alpha\beta}(f_\alpha) = 0$$

↳ Coulomb collisions conserve particle # (from Lec. #5)

and $f_\alpha \rightarrow 0$ as $v \rightarrow \infty$

$$\frac{\partial}{\partial t}\underbrace{\int d\vec{v} f_{\alpha_0}}_{n_{\alpha_0}} + \nabla\cdot\underbrace{\int d\vec{v}\,\vec{v} f_{\alpha_0}}_{n_{\alpha_0}\vec{u}_0} + \left(\frac{q}{m}\right)_\alpha \int d\vec{v}\,\underbrace{\overbrace{\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)}^{\vec{F}}\cdot\frac{\partial}{\partial\vec{v}} f_\alpha}_{\partial/\partial\vec{v}\cdot\vec{F} = 0} = \underbrace{\int d\vec{v}\,(C_{\alpha\alpha} + C_{\alpha\beta})}_{0}$$

$$\partial/\partial\vec{v}\cdot(\vec{E} + \vec{v}\times\vec{B}) = 0 + \partial/\partial\vec{v}\cdot(\vec{v}\times\vec{B})$$
$$= \vec{B}\cdot\underbrace{(\partial/\partial\vec{v}\times\vec{v})}_{0} = 0$$

$$\Rightarrow \frac{\partial n}{\partial t} + \nabla\cdot n\vec{u} = 0$$

The Continuity Equation

* implied sub$_0$ unless otherwise written from here on out

NOW: Momentum Equation

Operate with $\int d\vec{v}\,\vec{v}$ — multiply by $\vec{v}$ and integrate

→ Note: $\vec{v}$ is an independent variable and so passes

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through ∇, $\partial/\partial t$ operators (and $\int d\vec{v} f_{\alpha_0} = n_{\alpha_0}$)

$$\underbrace{\frac{\partial}{\partial t}\int d\vec{v} f_{\alpha_0}\vec{v}}_{n_0\vec{u}} + \nabla\cdot\int d\vec{v}\,\vec{v}\vec{v} f_{\alpha_0} + \int d\vec{v}\,\vec{v}\frac{\vec{F}}{m_\alpha}\cdot\frac{\partial}{\partial\vec{v}} f_{\alpha_1} = \underbrace{\int d\vec{v}\,\vec{v}\left(C_{\alpha\alpha_1} + C_{\alpha\beta_1}\right)}_{= \frac{-1}{m_\alpha}\vec{R}_{\alpha\beta_1}}$$

$$\int d\vec{v} f_\alpha \vec{v} = n_\alpha \vec{u}$$

where $\vec{R}_{\alpha\alpha} = 0$; i.e., no drag within a species

$$\overbrace{\frac{\partial}{\partial t} n_0\vec{u}}^{\text{local momentum}} + \underbrace{\overbrace{\nabla\cdot\int d\vec{v}\,\vec{v}\vec{v} f_{\alpha_0}}^{\text{flux of momentum}}}_{} - \overbrace{\frac{q_\alpha}{m_\alpha} n_\alpha\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right)}^{\text{change of momentum from } \vec{E}\text{-field}} = \underbrace{\frac{-1}{m_\alpha}\vec{R}_{\alpha\beta_1}}_{\text{momentum transfer to other species}}$$

we can extract average velocity from $\nabla\cdot\int d\vec{v}\,\vec{v}\vec{v} f_{\alpha_0}$

- Let $\vec{v} \equiv \vec{u} + \vec{v}'$
  - note that $\int dv' v' f_\alpha = 0$

$$\nabla\cdot\int d\vec{v}\,\vec{v}\vec{v} f_\alpha = \nabla\cdot(n_\alpha\vec{u}\vec{u}) + \nabla\cdot\int d\vec{v}'\,\vec{v}'\vec{v}' f_\alpha(\vec{v}')$$

$\Rightarrow$ Define the pressure tensor

$$\bar{\bar{P}}_\alpha = m_\alpha\int d\vec{v}'\,\vec{v}'\vec{v}' f_\alpha$$

Then, rearranging the above equation

$$\frac{\partial}{\partial t} n_\alpha\vec{u} + \nabla\cdot\int d\vec{v}\,\vec{v}\vec{v} f_\alpha = \left(\frac{q}{m}\right)_\alpha n_\alpha\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right) - \frac{1}{m_\alpha}\vec{R}_{\alpha\beta_1}$$

$$m_\alpha\Big(\frac{\partial}{\partial t} n_\alpha\vec{u} + \underbrace{\nabla\cdot\int d\vec{v}\,\vec{v}\vec{v} f_\alpha}_{\text{plug in value defined above}}\Big) = q_\alpha n_\alpha\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right) - \vec{R}_{\alpha\beta_1}$$

$$m_\alpha\left[\frac{\partial}{\partial t} n_0\vec{u} + \nabla\cdot(n_0\vec{u}\vec{u}) + \nabla\cdot\int d\vec{v}'\,\vec{v}'\vec{v}' f_\alpha(\vec{v}')\right] = q_\alpha n_\alpha\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right) - \vec{R}_{\alpha\beta_1}$$

$$m_\alpha\left[\frac{\partial}{\partial t} n_0\vec{u} + \nabla\cdot(n_0\vec{u}\vec{u})\right] + \nabla\cdot\underbrace{m_\alpha\int d\vec{v}'\,\vec{v}'\vec{v}' f_\alpha(\vec{v}')}_{\bar{\bar{P}}_\alpha} = q_\alpha n_\alpha\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right) - \vec{R}_{\alpha\beta_1}$$

$$m_\alpha\left[\frac{\partial}{\partial t} n_0\vec{u} + \nabla\cdot(n_0\vec{u}\vec{u})\right] = q_\alpha n_\alpha\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right) - \nabla\cdot\bar{\bar{P}}_\alpha - \vec{R}_{\alpha\beta_1}$$

use the continuity equation, $\frac{\partial}{\partial t} n_0 + \nabla\cdot n_0\vec{u} = 0$, to extract n_0 on this LHS

$$n_0 m_\alpha\Big[\underbrace{\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\nabla\vec{u}}_{\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla}\Big] = n_\alpha q_\alpha\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right) - \nabla\cdot\bar{\bar{P}}_\alpha - \vec{R}_{\alpha\beta_1}$$

$\longrightarrow$

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$$n_0 m_\alpha \frac{d}{dt}\vec{u} = \underbrace{n_\alpha q_\alpha\left(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}\right)} - \underbrace{\nabla\cdot\overline{\overline{P_\alpha}}} - \vec{R}_{\alpha\beta,}$$

Must break this down into rank 1 & rank ε

$$\nabla\cdot\overline{\overline{P_\alpha}} = \nabla P_\alpha + \underbrace{\nabla\cdot\overline{\overline{\delta P_\alpha}}}_{\text{viscous stress}} \approx \nabla P_\alpha$$

$\sim 1/\nu \to$ small

take $\vec{E},\vec{B}$ 1st order; all other terms in 0th order

$$n_0 m_\alpha \frac{d}{dt}\vec{u} = -\nabla P_\alpha - \vec{R}_{\alpha\beta,1} + n_{\alpha_0} q_\alpha\left(\vec{E}_1 + \frac{1}{c}\vec{u}\times\vec{B}_1\right) + q_\alpha\left(n_{\alpha_1}\vec{E}_0 + \frac{1}{c}\underbrace{(n\vec{u})_{\alpha_1}}_{1/q_\alpha\,\cdot\,\text{first-order current, }\vec{J}_1}\times\vec{B}_0\right)$$

↓ sum over species α and β ↓

$n_{e_0} = n_{i_0}$; 0th ⊙ Ampère's Law

0; symmetry properties

$$n_0(m_\alpha + m_\beta)\frac{d}{dt}\vec{u} = -\nabla(P_\alpha + P_\beta) - \cancel{(\vec{R}_{\alpha\beta,1} + \vec{R}_{\beta\alpha,1})} + n_{\alpha_0}\cancel{(-e+e)}\left(\vec{E}_1 + \frac{1}{c}\vec{u}\times\vec{B}_1\right) + \cancel{(-e+e)}(n_{\alpha_1}\vec{E}_0) + \frac{1}{c}\vec{J}_1\times\vec{B}_0$$

0 0

$$\vec{J}_1 = e(n\vec{u})_{i_1} - e(n\vec{u})_{e_1}$$

$$n(m_\alpha + m_\beta)\frac{d}{dt}\vec{u} = -\nabla(P_\alpha + P_\beta) + \frac{1}{c}\vec{J}_1\times\vec{B}$$

*Define:

$\rho \equiv (m_\alpha + m_\beta)n$, total 0th order density

$P \equiv P_\alpha + P_\beta$, total pressure

↓ plugging in ↓

$$\rho\frac{d}{dt}\vec{u} = -\nabla P + \frac{1}{c}\vec{J}_1\times\vec{B}$$ The Momentum Equation

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}_1$$

⇒ We now know n_0, u_0/E_0!

Only need an expression for the pressure now

Pressure Equation

Operate with $\int d\vec{v}\, v^2$ — multiply by $\frac{1}{2}mv^2$ and integrate

$$\frac{\partial}{\partial t}\int d\vec{v}\,\frac{m_\alpha v^2}{2} f_\alpha + \nabla\cdot\left[\int d\vec{v}\,\frac{m_\alpha v^2}{2}\vec{v} f_\alpha\right] + \frac{q_\alpha}{\cancel{m_\alpha}}\int d\vec{v}\,\frac{\cancel{m_\alpha} v^2}{2}\frac{\partial}{\partial\vec{v}}\cdot\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)f_\alpha$$

$$= \int d\vec{v}\,\frac{m_\alpha v^2}{2}\left(C_{\alpha\alpha} + C_{\alpha\beta}\right)$$

Recall f_α is a drifting Maxwellian $f_\alpha = \frac{n_\alpha}{(2\pi T_\alpha/m_\alpha)^{3/2}} e^{-(\vec{v}-\vec{u})^2 m_\alpha/2T_\alpha}$

$\vec{v}' = \vec{v} - \vec{u} \longrightarrow \vec{v} = \vec{v}' + \vec{u}$

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Plugging in term-by-term:

- $\int d\vec{v}\,\frac{m_\alpha v^2}{2} f_\alpha = \frac{m_\alpha u^2}{2}\underbrace{\int d\vec{v} f_\alpha}_{n} + \frac{\cancel{m_\alpha}}{2}\underbrace{\int d\vec{v}'\, v'^2 f_\alpha}_{3nT_\alpha/\cancel{m_\alpha}}$ — cross-terms go away: odd in $\vec{v}'=0$

$$= \tfrac{1}{2} m_\alpha u^2 n + \tfrac{3}{2} n T_\alpha$$

- $\int d\vec{v}\,\frac{m_\alpha v^2}{2}\vec{v} f_\alpha = \int d\vec{v}\,\frac{m_\alpha(\vec{v}'+\vec{u})^2}{2}(\vec{v}'+\vec{u}) f_\alpha$

$$= \frac{m_\alpha u^2}{2}\vec{u}\underbrace{\int d\vec{v} f_\alpha}_{n} + \underbrace{\int d\vec{v}'\,\frac{m_\alpha v'^2}{2}\vec{v}' f_\alpha}_{\equiv \vec{Q}_\alpha} + \frac{\cancel{m_\alpha}}{2}\vec{u}\underbrace{\int d\vec{v}'\, v'^2 f_\alpha}_{3nT_\alpha/\cancel{m_\alpha}}$$

$$+ \int d\vec{v}'\,\frac{\cancel{m_\alpha}}{\cancel{2}}\, 2\underbrace{\vec{v}'\cdot\vec{u}\,\vec{v}'}\, f_\alpha$$

only survives if two components of $\vec{v}'$ are the same
↳ lose two degrees of freedom
$\int d\vec{v}'\, v'^2 f_\alpha - 2$ D.O.F. $= nT_\alpha/\cancel{m_\alpha}$

$$= \tfrac{1}{2} n m_\alpha u^2 \vec{u} + \vec{Q}_\alpha + \tfrac{3}{2} n T_\alpha \vec{u} + n T_\alpha \vec{u}$$

heat flux

$$= \left(\tfrac{1}{2} n m_\alpha u^2 + \tfrac{3}{2} n T_\alpha\right)\vec{u} + \vec{Q}_\alpha + n T_\alpha \vec{u}$$

- $q_\alpha \int d\vec{v}\,\underbrace{\frac{v^2}{2}\frac{\partial}{\partial\vec{v}}}_{-\vec{v}}\cdot\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right) f_\alpha = -q_\alpha\int d\vec{v}\,\vec{v}\cdot\left(\vec{E}+\cancelto{0}{\frac{1}{c}\vec{v}\times\vec{B}}\right) f_\alpha$

motion of fluid in direction of field

$$= -q_\alpha \vec{E}\underbrace{\int d\vec{v}\,\vec{v} f_\alpha}_{\vec{u} n_\alpha} = \underbrace{-q_\alpha \vec{E}\cdot\vec{u} n_\alpha}$$

must break this down into rank 1 and rank ε

rank ε n for rank 1 $\vec{E}$

$$= -q_\alpha\left[n\vec{u}\cdot\vec{E}_1 + \underbrace{(n\vec{u})_{\alpha_1}}_{1/q_\alpha \vec{J}_1}\cdot\vec{E}_0\right]$$

→

Putting it all together:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}m_\alpha u^2 n + \frac{3}{2}\underbrace{nT_\alpha}_{P_\alpha}\right) + \nabla\cdot\left(\frac{1}{2}n m_\alpha u^2 + \frac{3}{2}\underbrace{nT_\alpha}_{P_\alpha}\right)\vec{u} + \overbrace{\nabla\cdot\vec{Q}_\alpha}^{\text{usually assumed small}} + \nabla\cdot\left(\underbrace{nT_\alpha}_{P_\alpha}\vec{u}\right)$$

$$= q_\alpha\left[n\vec{u}\cdot\vec{E}_1 + \frac{\vec{J}_1}{q_\alpha}\cdot\vec{E}_0\right] + \underbrace{\sum_\beta\int d\vec{v}\,\frac{m_\alpha v^2}{2}\left(C_{\alpha\alpha}+C_{\alpha\beta}\right)}_{= -\cancelto{0}{\left(\frac{\partial W}{\partial t}\right)_{\alpha\alpha}} - \left(\frac{\partial W}{\partial t}\right)_{\alpha\beta}}$$

from C symmetry

$$\frac{\partial}{\partial t}\left(\frac{1}{2}m_\alpha n u^2 + \frac{3}{2}P_\alpha\right) + \nabla\cdot\left(\frac{1}{2}n m_\alpha u^2 + \frac{3}{2}P_\alpha\right)\vec{u} + \nabla\cdot\left(P_\alpha\vec{u}\right)$$

$$= q_\alpha n\vec{u}\cdot\vec{E}_1 + \vec{J}_1\cdot\vec{E}_0 - \left(\frac{\partial W}{\partial t}\right)_{\alpha\beta}$$

↓ sum over species α and β ↓

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\underbrace{(m_\alpha+m_\beta)n}_{\text{total mass density, }\rho}u^2 + \frac{3}{2}\overbrace{(P_\alpha+P_\beta)}^{\text{total pressure, }P}\right] + \nabla\cdot\left[\frac{1}{2}\underbrace{n(m_\alpha+m_\beta)}_{\rho}u^2 + \frac{3}{2}\overbrace{(P_\alpha+P_\beta)}^{P}\right]\vec{u} + \nabla\cdot\left[\overbrace{(P_\alpha+P_\beta)}^{P}\vec{u}\right]$$

$$= \cancelto{0}{(-e+e)}n\vec{u}\cdot\vec{E}_1 + \vec{J}_1\cdot\vec{E}_0 - \cancelto{0}{\left[\left(\frac{\partial W}{\partial t}\right)_{\alpha\beta} + \left(\frac{\partial W}{\partial t}\right)_{\beta\alpha}\right]}$$

0; conservation of energy

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right) + \nabla\cdot\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)\vec{u} + \nabla\cdot\left(P\vec{u}\right)$$

$$= \vec{J}_1\cdot\vec{E} = -\frac{1}{c}\vec{J}_1\cdot\left(\vec{u}\times\vec{B}\right)$$

plug in $\vec{E}_0 = -\frac{1}{c}\vec{u}\times\vec{B}$

$$= \vec{u}\cdot\frac{1}{c}\left(\vec{J}_1\times\vec{B}\right)$$

flipping order flips sign: $\vec{A}\cdot\vec{B}\times\vec{C} = -\vec{B}\cdot(\vec{A}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

permutations do not change sign

then from the momentum equation

$$\frac{1}{c}\vec{J}_1\times\vec{B} = \rho\frac{d}{dt}\vec{u} + \nabla P$$

↓ plugging in ↓

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right) + \nabla\cdot\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)\vec{u} + \nabla\cdot\left(P\vec{u}\right) = \vec{u}\cdot\left[\rho\frac{d}{dt}\vec{u} + \nabla P\right]$$

The Energy Equation

*$\vec{B}$ has completely dropped out! This yields the same result as before with our fluid eqns., even with a $\vec{B}$-field present!

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$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2+\tfrac{3}{2}P\right)+\nabla\cdot\left(\tfrac{1}{2}\rho u^2+\tfrac{3}{2}P\right)\vec{u}+\nabla\cdot(P\vec{u})=\vec{u}\cdot\left[\rho\frac{d}{dt}\vec{u}+\nabla P\right]$$
$$=\vec{u}\cdot\nabla P+\underbrace{\rho\vec{u}\cdot\frac{d}{dt}\vec{u}}_{=\rho\frac{d}{dt}\frac{u^2}{2}}$$

$$\uparrow\ \frac{d}{dt}=\frac{\partial}{\partial t}+\vec{u}\cdot\nabla$$

$$\frac{1}{2}u^2\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial t}\left(\tfrac{3}{2}P\right)+\vec{u}\cdot\nabla\left(\tfrac{1}{2}\rho u^2+\tfrac{3}{2}P\right)+\left(\tfrac{1}{2}\rho u^2+\tfrac{3}{2}P\right)\nabla\cdot\vec{u}+\cancel{\vec{u}\cdot\nabla P}$$
$$+P\nabla\cdot\vec{u}=\cancel{\vec{u}\cdot\nabla P}+\underbrace{\rho\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)\frac{u^2}{2}}_{\rho\left(\cancel{\frac{\partial}{\partial t}u^2/2}\right)+\rho(\vec{u}\cdot\nabla)\frac{u^2}{2}}$$

$\to 0$ $\frac{\partial}{\partial t}$ does not operate on u^2 as seen in first term of LHS

$$\frac{1}{2}u^2\left(\frac{\partial\rho}{\partial t}+(\vec{u}\cdot\nabla)\rho+\rho(\nabla\cdot\vec{u})\right)+\frac{\partial}{\partial t}\left(\tfrac{3}{2}P\right)+\vec{u}\cdot\nabla\left(\tfrac{3}{2}P\right)+\left(\tfrac{3}{2}P\right)\nabla\cdot\vec{u}$$
$$+P\nabla\cdot\vec{u}=\rho(\vec{u}\cdot\nabla)\frac{u^2}{2}$$
$$=\frac{1}{2}u^2(\vec{u}\cdot\nabla)\rho$$

move to LHS; expand $\partial\rho/\partial t$

$$-\frac{1}{2}u^2\left[\cancel{\rho(\nabla\cdot\vec{u})}+\cancel{(\vec{u}\cdot\nabla)\rho}\right]+\frac{\partial}{\partial t}(3/2P)+\cancel{\tfrac{1}{2}u^2(\vec{u}\cdot\nabla)\rho}+\frac{3}{2}\vec{u}\cdot\nabla P+\left(\cancel{\tfrac{1}{2}\rho u^2}+\tfrac{5}{2}P\right)\nabla\cdot\vec{u}=0$$

$$\underbrace{\frac{\partial}{\partial t}\left(\tfrac{3}{2}P\right)+\tfrac{3}{2}\vec{u}\cdot\nabla P}_{\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{u}\cdot\nabla}+\left(\tfrac{5}{2}P\right)\nabla\cdot\vec{u}=0$$

$$\Rightarrow\frac{d}{dt}P+\frac{5}{3}P\nabla\cdot\vec{u}=0$$

Summarizing...

The Magnetohydrodynamics Equations

→ Pressure/Energy Equation:

$$\frac{d}{dt}P+\frac{5}{3}P\nabla\cdot\vec{u}=0$$ evolves P

→ Continuity Equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\vec{u}=0$$ evolves mass density

ρ varying; $\rho=(m_i+m_e)n$

→

→ Momentum Equation:

$$\rho \frac{d}{dt}\vec{u} = -\nabla P + \frac{1}{c}\vec{J}\times\vec{B}$$ evolves $\vec{u}$

$\vec{J} = \vec{J}_i$, but we drop the index

$$= -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B}$$

$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}$

$\nabla\cdot\vec{B} = 0$

$\vec{E} = -\frac{1}{c}\vec{u}\times\vec{B}$

*Note: take $\vec{E}\cdot\vec{B}$

↳ $E_{\parallel} = \frac{\vec{E}\cdot\vec{B}}{B} = 0$ ⇒ no parallel electric field in traditional MHD model

$\frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \nabla\times\vec{E} = 0$ evolves $\vec{B}$

NOW: Using the MHD equations

↳ starting with "ground zero"

Ideal MHD Equilibria

In exploring the stability & dynamics of magnetized plasma, it is useful to start with states that are time stationary so that they are in a state of equilibrium. We will discuss several examples of such states.

↳ for simplicity, we will limit this discussion to situations without flow, although in many cases equilibria with flow can also be constructed

- spatial- but no time dependence (plasma in a state of equilibrium)
- no waves
- no instabilities
- $\vec{u} = 0$ (no flow)

} Stationary Systems → want to know what these look like

→

The basic MHD equations:

① $0 = -\nabla P + \frac{1}{c}\vec{J}\times\vec{B}$

$\quad\llcorner\ \vec{J} = \frac{c}{4\pi}(\nabla\times\vec{B})$

$= -\nabla P + \frac{1}{c}\frac{c}{4\pi}\underbrace{(\nabla\times\vec{B})\times\vec{B}}_{= -\frac{1}{2}\nabla B^2 + \vec{B}\cdot\nabla\vec{B}}$

$= -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B}$

② $\vec{E} = 0$

③ $\nabla\cdot\vec{B} = 0$

④ $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}$

What do these tell us?

- Take $\vec{B}\cdot$ eqn. ① $\rightarrow \vec{B}\cdot\nabla P = 0$
  - $\Rightarrow$ Pressure must be constant along $\vec{B}$-field lines
  - * non-constant pressure generates a sound wave that smooths/relaxes parallel pressure gradients out!
- Take $\nabla\cdot$ eqn. ④ $\rightarrow \nabla\cdot(\nabla\times\vec{B}) = 0 = \nabla\cdot\left(\frac{4\pi}{c}\vec{J}\right)$

  $\nabla\cdot\vec{J} = 0$
  - $\Rightarrow$ There is no build-up of charge
  - * charge build-up would produce an $\vec{E}\neq 0$, and since $\vec{E} = -\frac{1}{c}\vec{u}\times\vec{B}$, flows would be generated, which we have forbidden

Consider some examples...

Case I: $\underbrace{(\vec{B}\cdot\nabla)\vec{B} = 0}$

means we have straight field lines

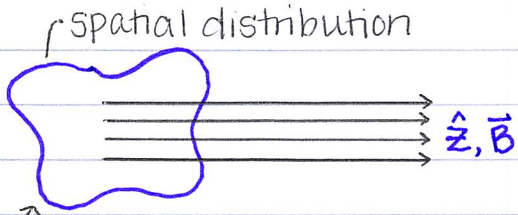
($\hat{b}$ and $|\vec{B}|$ unchanging)

$\Rightarrow \nabla\left(P + \frac{B^2}{8\pi}\right) = 0$, Total pressure must be a constant

plasma pressure (P) magnetic pressure ($\frac{B^2}{8\pi}$)

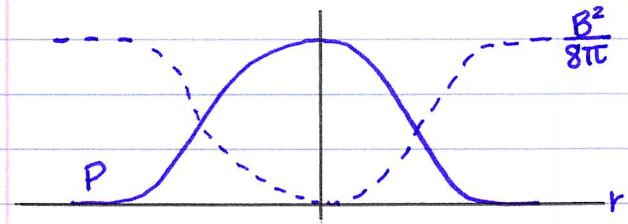
How does this impact the spatial distributions of P and B in our system?

KLE



You can have any spatial distribution of P & B that you want as long as $P + \frac{B^2}{8\pi} = \text{constant}$

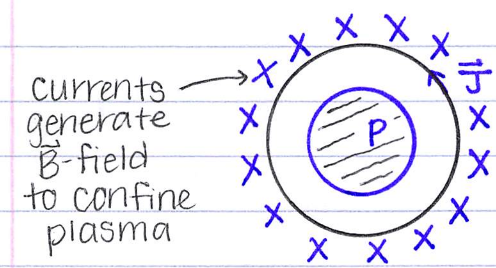
- This is not a trivial state ↳ e.g., in 1-D:



*this is a localized plasma system supported by a $\vec{B}$-field on the outside

Confinement experiments with such equilibria are called θ-pinches

$$\frac{\partial}{\partial r}\left(P + \frac{B^2}{8\pi}\right) = 0$$



azimuthal currents (in $\hat{\theta}$ direction)

↳ want azimuthal component of ①

$$-\underbrace{\frac{\partial}{\partial r} B_z}_{\text{radial dependence on } B_z} = \frac{4\pi}{c} J_\theta$$

<u>Case II</u> : $(\vec{B}\cdot\nabla)\vec{B} \neq 0$

↑ we now have curved field lines

Simple case —

$\vec{B} = B_\theta \hat{\theta}$; azimuthal $\vec{B}$-field only

then

$$(\vec{B}\cdot\nabla)\vec{B} = [(\underbrace{\hat{b}|\vec{B}|}_{\hat{b} \equiv \frac{\vec{B}}{|\vec{B}|}})\cdot\nabla](\hat{b}|\vec{B}|)$$

$$= B^2(\hat{b}\cdot\nabla)\hat{b} = B^2\vec{\kappa}$$

recall: $\vec{\kappa} \equiv \hat{b}\cdot\nabla\hat{b}$, the curvature of $\vec{B}$

$$= \frac{B_\theta}{r}\frac{\partial}{\partial\theta} B_\theta \hat{\theta}$$

↓ take B_θ to be independent of θ ↓

$$(\vec{B}\cdot\nabla)\vec{B} = \frac{B_\theta}{r} B_\theta \underbrace{\frac{\partial}{\partial\theta}\hat{\theta}}_{-\hat{r}} = \underbrace{\frac{-B_\theta^2}{r}\hat{r}}_{\text{compare back to } B^2\vec{K}}$$

where $K \sim 1/R_c \Rightarrow R_c = r$

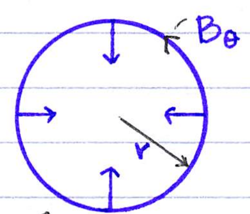
└ radius of curvature

eqn. ① becomes

$$\frac{\partial}{\partial r}\left(P + \frac{B_\theta^2}{8\pi}\right) + \underbrace{\frac{B_\theta^2}{r}}_{\text{magnetic tension}} = 0$$

↳ Tension force from the $\vec{B}$-field can help balance variations in the total plasma pressure (that may not be uniform in space)

⇒ Field can confine a plasma!



J_z out of the page — current along $\hat{z}$ is a z-pinch

$$(\nabla\times(B_\theta\hat{\theta}))_z = \frac{4\pi}{c}J_z$$

└ curl of $\hat{\theta} \neq 0$! → write $\hat{\theta}$ as $r\nabla\theta$

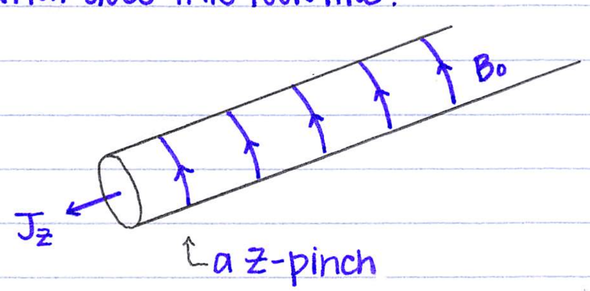
$$(\nabla\times rB_\theta\underbrace{\nabla\theta}_{})_z = \frac{4\pi}{c}J_z$$

$\nabla\times\nabla\theta = 0$, so we no longer have to take the curl of this

$$[\underbrace{\nabla(rB_\theta)}_{\frac{\partial}{\partial r}(rB_\theta)}\times\underbrace{\nabla\theta}_{\frac{1}{r}}]_z = \frac{4\pi}{c}J_z$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}rB_\theta = \frac{4\pi}{c}J_z$$

What does this look like?



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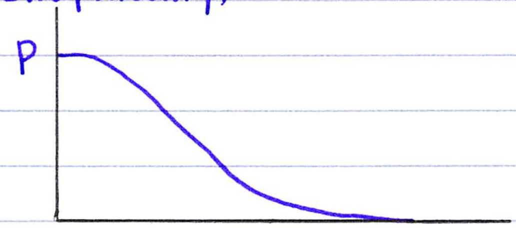
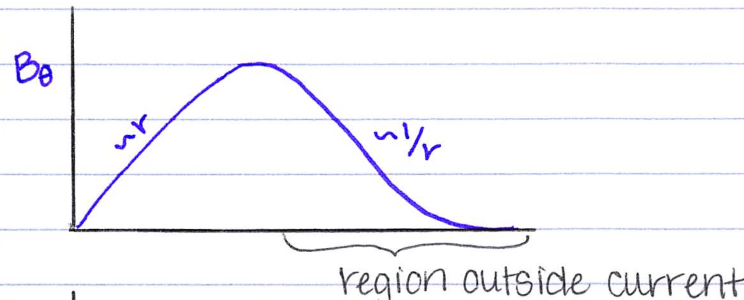
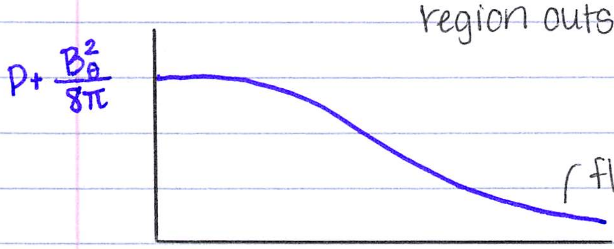
Graphically,

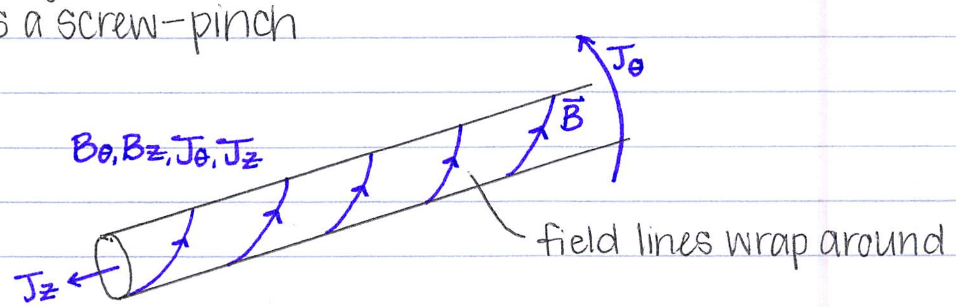






flattens out much slower than P due to contribution from B_θ

Experiments with both $B_\theta, B_z \neq 0$ are called screw-pinches.
↳ having B_θ & B_z produces J_z & J_θ
* Magnetic reconnection in the solar corona may be modeled as a screw-pinch



** If you bend this into a torus, really crazy stuff happens!