Lecture 17 – General Particle Drifts 10/24/17

~~[illegible]~~

General Particle Drifts

No $\partial B / \partial t$

→ weak space variation ~~[illegible]~~

$\underbrace{\frac{v_\perp}{\Omega}}$ $\ll L$ ~~[illegible]~~

local Larmor radius, $r_L = v_\perp / \Omega$

→ $E \ll B \Rightarrow \underbrace{\vec{v}_E}_{cE/B} \ll c$, $\vec{v}_E = c\vec{E} \times \vec{B} / B^2$ (E×B drift)

Where (slow time scale)

$X(t) = \vec{X}_{gc} + \vec{r}_L(t)$

└ fast time scale

$v(t) = \vec{v}_{gc} + \vec{v}_L(t)$

└ velocity assoc. w/ larmor rad. (fast gyro motion)

We showed from this:

$m\dot{\vec{v}}_{gc} = q\vec{E} + \frac{q}{c}\vec{v}_{gc} \times \vec{B} - \mu \nabla B$ eq. (1)

dropping notation

$\mu = \frac{m v_\perp^2}{2B}$ mag moment

HW #7 Prob 1

→ expand about resonance

- real part of $\varepsilon \to 0$ → will get Lorentzian about both acoustic & pl
- integral ~~[illegible]~~ indep. of what $\mathrm{Im}\{\varepsilon\}$ actually is — don't need to evaluate

$L^3 \left\langle \frac{|E|^2}{8\pi} \right\rangle n(\ ) \, \mathrm{Im}\left\{\frac{1}{\varepsilon_{k\omega}}\right\}$

NOW: Split $\vec{v}_{gc} = \vec{v}$* (slow time-var) into parallel and perpendicular components

want to find this

$\vec{v} = v_\parallel \hat{b} + \vec{v}_\perp$

mag field unit vec. ↑ (B field twisting around in some way)

slow drift assoc. w/ B-field, NOT fast gyro motion

*Note that the unit vector is time-dep.

$\dot{\vec{v}} = \frac{d}{dt}(v_\parallel \hat{b} + \vec{v}_\perp)$ ~~[illegible]~~

$$\dot{\vec{v}} = \frac{d}{dt}\vec{v}_\perp + \hat{b}\frac{dv_\parallel}{dt} + v_\parallel \frac{d}{dt}\hat{b}$$

Along $\hat{b}$ - ~~insert~~ above into original eqn; dot w/ $\hat{b}$ ~~[illegible]~~ dropped

$$\underbrace{m\hat{b}\cdot\frac{d}{dt}\vec{v}_\perp}_{\text{switch order}} + m\frac{dv_\parallel}{dt} + \cancel{v_\parallel \underbrace{\hat{b}\cdot\frac{d}{dt}\hat{b}}_{d/dt|\hat{b}|^2}}^{0} = q\hat{b}\cdot\vec{E} - \mu\hat{b}\cdot\nabla\vec{B}$$

$= -m\vec{v}_\perp \cdot \frac{d}{dt}\hat{b}$

$= m v_\perp v_t (1/L)$

$\Rightarrow$ take $v_\perp << v_t \rightarrow$ throw out this term; small

$= d/dt(1)$, $\hat{b}$ a unit vector

$= 0$

$-\mu\hat{b}\cdot\nabla\vec{B} \rightarrow \frac{mB}{L}\frac{v_\perp^2}{2B} = \frac{v_t^2}{2\cancel{B}}\frac{\cancel{B}}{L}m$

$\Rightarrow \frac{mv_t}{L}$

$\Rightarrow$ operate in limit where

$\vec{v}_E << v_t$

then

$$m\frac{dv_\parallel}{dt} = qE_\parallel \underbrace{- \mu\hat{b}\cdot\nabla B}_{\text{repulsive term}}$$

looking at the perpendicular component:

☆ $\frac{d}{dt}\hat{b} = v_\parallel \underbrace{\hat{b}\cdot\nabla\hat{b}} + \cancel{v_\perp\cdot\nabla\hat{b}} \rightarrow \nabla\hat{b}\cdot\hat{b} = 0$
$= \hat{b}\cdot\nabla\hat{b} = 0$ what?

define $\vec{K} \equiv \hat{b}\cdot\nabla\hat{b}$, the curvature of $\vec{B}$
└ curvature vector

$= v_\parallel \vec{K}$

$\hat{b}\cdot\vec{K} = 0 = \hat{b}\cdot\hat{b}\cdot\nabla\hat{b}$

└ $\hat{b}$ perp to curvature vector

Returning to eq. ① - focus on perp. comp.

☆ $m\frac{d}{dt}\vec{v}_\perp + mv_\parallel\vec{K} = q\vec{E}_\perp + \frac{q}{c}\vec{v}_\perp\times\vec{B} - \mu\nabla_\perp B$ $\quad\nabla_\perp$?

combine into perpendicular force

*slow time var wrt cycl. freq

$\vec{F}_\perp = q\vec{E}_\perp - \mu \nabla_\perp B - m v_\parallel \vec{\kappa}$ — can iterate to lowest order — no need to keep higher terms

$\hookrightarrow \underbrace{m \frac{d}{dt} \vec{v}_\perp}_{m\omega \vec{v}_\perp} = \vec{F}_\perp + \underbrace{\frac{q}{c} \vec{v}_\perp \times \vec{B}}_{q/cm \vec{v}_\perp}$ what are dom. terms?

★ ? couldn't read

$\Omega v_\perp \rightarrow$ gyro freq

toss this term! gyro freq dominant!

$\rightarrow \vec{B} \times \left(\frac{q}{c} \vec{v}_\perp \times \vec{B} = -\vec{F}_\perp \right)$

$\frac{q}{c} \vec{v}_\perp B^2 = -\vec{B} \times \vec{F}_\perp$

$\hookrightarrow \vec{v}_{\perp 0} = \frac{c}{qB^2} \vec{F}_\perp \times \vec{B}$

throw out higher terms

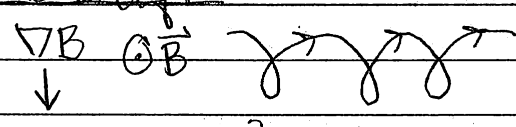
$= \frac{c\vec{E} \times \vec{B}}{B^2} - \frac{c}{qB} \left(\mu \nabla B \times \hat{b} + m v_\parallel^2 \vec{\kappa} \times \hat{b} \right)$

grad curv

$\Rightarrow v_{\nabla B} = \frac{c}{qB} \mu \hat{b} \times \nabla B$

★ gradient B drift perp to what?

<u>∇B Drift</u>



★ $\Rightarrow v_c = \frac{v_\parallel^2}{\Omega} \hat{b} \times \vec{\kappa}$ curvature drift what?

inspect scaling:

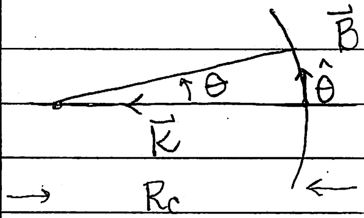
$v_{\nabla B} \sim \frac{c}{q\cancel{B}} \frac{\cancel{m} v_\perp^2}{2\cancel{B}\Omega} \frac{\cancel{B}}{L} \sim v_t \frac{\rho_L}{L}$ — $\rho_L = \frac{v_\perp}{\Omega}$ larmor rad

Drift small compared to the thermal velocity!

$v_c \sim \frac{v_t^2 \, mc}{qB\Omega \, R} \sim v_t \frac{\rho_L}{R}$ appx. same scaling as $v_{\nabla B}$

★ R = radius of curvature? R vs L?

NOW: Look at $\vec{K}$



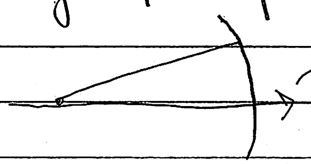
$\vec{K} = \hat{b}\cdot\nabla\hat{b}$

$\underbrace{\frac{1}{R}\frac{\partial}{\partial\theta}}_{-\hat{r}}\hat{\theta} = -\frac{1}{R_C}\hat{r}$

$|\vec{K}|$

└ radius of curvature of magnetic field lines

What, then, is curvature drift?
└ jump to particle ref. frame



feels centrifical force (effective force)

Effective force $\times\,\hat{b}$ gives curvature drift

$\vec{F}_c = -mv_{\parallel}^2\vec{K} \Rightarrow \frac{c}{qB^2}\vec{F}_c\times\vec{B}$ in particle frame

Recap:
Impt. drifts: polarization drift?
1. $E\times B$
2. ∇grad
3. curv.

Note: As the particle drifts around, μ is a conserv. quantity
Conservation of μ
- slowly var. fields
→ BUT $\frac{\partial B}{\partial t} \neq 0$! where $\mu = \frac{mv_\perp^2}{2B}$

μ └ conserved even if B is changing!

$m\frac{d}{dt}v_{\parallel} = qE_{\parallel} - \mu\hat{b}\cdot\nabla B$
└ want to construct energy eq.

Parallel energy equation:

$$v_\parallel \left(m\frac{d}{dt} v_\parallel = qE_\parallel \qquad \right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} m v_\parallel^2\right) = q v_\parallel E_\parallel - v_\parallel \mu \hat{b}\cdot\nabla B$$

Want perp eqn → can NOT transform E field away!

$$\oint \vec{E}\cdot d\vec{l} \neq 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \nabla\times\vec{E} = 0$$

cannot move into E×B drift frame when B changes in time

→ construct energy eqn from orig. EOM

orig. EOM

$$\vec{v}\cdot\left(m\frac{d}{dt}\vec{v} = q\vec{E} + \frac{1}{c}\vec{v}\times B\right)$$

$$\frac{d}{dt}\frac{mv^2}{2} = q\vec{E}\cdot\vec{v}$$

dom by parallel streaming & larmor motion

($v_\perp$ small → throw out)

$$\frac{d}{dt}\left(\frac{1}{2}m v_\parallel^2 + \underbrace{\frac{1}{2}m v_\perp^2}_{\text{larmor KE}}\right) = q v_\parallel E_\parallel + q\vec{E}_\perp\cdot\vec{v}_\perp$$

$$+\cancel{\frac{1}{2}m v_\perp^2} \to 0$$

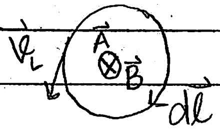
then

$$\frac{d}{dt}\left(\frac{1}{2}m v_\perp^2\right) = \underbrace{q\vec{v}_\perp\cdot\vec{E}_\perp}_{\text{need to deal w/ this term}} + v_\parallel \mu \hat{b}\cdot\nabla B$$

↓ avg. this eqn over on larmor period / orbit ↓

$$\langle \vec{v}_\perp\cdot\vec{E}_\perp\rangle = \frac{\Omega}{2\pi}\int_0^\tau dt\, \vec{v}_\perp\vec{E}_\perp, \quad \tau = \text{orbital period}$$

time avg. (took avg over whole eqn., but everything else cancels out)



→ $\vec{v}_\perp$ vector opposite direction of $d\vec{l}$ $\quad \vec{v}_\perp dt = -d\vec{l}$

$\vec{A}$ = area vector

$= -\frac{\Omega}{2\pi}\int d\vec{l}\cdot E_\perp$

↓ perform Stokes' thm ↓

$\int d\vec{l}\cdot E_\perp = \int_A d\vec{A}\cdot \underbrace{\nabla\times\vec{E}}$

$\frac{1}{c}\,\partial B/\partial t + \nabla\times\vec{E} = 0$

$= +\frac{\Omega}{2\pi}\frac{1}{c}\int d\vec{A}\cdot\frac{\partial\vec{B}}{\partial t}$ in same direction

area of orbit

$= +\frac{\Omega}{2\cancel{\pi}}\frac{1}{c}(\cancel{\pi} r_L^2)\frac{\partial}{\partial t}B$

$= \frac{\cancel{\Omega}}{2c}\frac{v_\perp^2}{\Omega^{\cancel{2}}}\frac{\partial}{\partial t}B = \frac{v_\perp^2 m\cancel{c}}{2\cancel{c}qB}\frac{\partial B}{\partial t} = \frac{\mu}{q}\frac{\partial B}{\partial t}$

then

$\frac{d}{dt}\left[\frac{1}{2}mv_\perp^2\left(\frac{B}{B}\right)\right] = q\vec{v}_\perp\cdot\vec{E}_\perp + v_\parallel\mu\hat{b}\cdot\nabla\vec{B}$

insert plug in

$\underbrace{\frac{d}{dt}(\mu B)} = \mu\left(\frac{\partial B}{\partial t} + v_\parallel\hat{b}\cdot\nabla B\right)$

$\hookrightarrow \frac{d}{dt}(\mu B) = \mu\frac{dB}{dt}$

$B\frac{d\mu}{dt} + \cancel{\mu\frac{dB}{dt}} = \cancel{\mu\frac{dB}{dt}} = \frac{d\mu}{dt} = 0 \Rightarrow$ μ is conserved

$\Rightarrow$ if you have particles drifting, can determine local $\vec{B}$ (b/c know 3 drifts and $v_\parallel$), then you can get $\vec{v}_\perp$ using μ conservation (no separate determination for $\vec{v}_\perp$)

$v_\parallel$ constant over larmor radius $\longrightarrow$

μ conservation $\rightarrow$ allows us to show,

canon. momentum

$J = \oint p\,dq = 2\pi P_\theta$

$\llcorner P_\theta$ = canonical ang. mom

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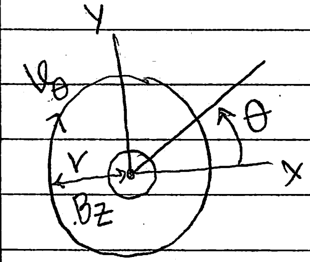

r= larmor orbit radius

must calc vect. pot. in what direction?? A_θ

$$B_z = \hat{z}\cdot\nabla\times\vec{A} = \frac{1}{r}\frac{\partial}{\partial r} r A_\theta$$

$$\Rightarrow A_\theta = B_z r/2$$

from this:

$$P_\theta = m r \underbrace{v_\theta}_{r\dot{\theta}} + \frac{q}{c} A_\theta r$$

plug in

$$P_\theta = m r^2 \dot{\theta} + \frac{qB_z}{2c} r^2$$

$$\dot{\theta} = -\Omega$$

$$= -mr^2\Omega + \frac{qB_z}{2c} r^2 = -mr^2\frac{qB_z}{mc} + \frac{qB_z}{2c} r^2 = -\frac{mr^2 q B_z}{2mc}$$

plug in

$$= -\frac{m}{2}\frac{v_\perp^2}{\Omega^2}\Omega = -\frac{m}{2}\frac{v_\perp^2}{qB} mc = -\frac{mc}{q}\mu$$

mag. moment

$\Rightarrow$ Canon. ang. mom. directly related to mag. moment

Identity for vacuum mag. field

* vacuum $\vec{B}$ → no current *

$\Rightarrow \vec{J} = 0$

then

what is this?

$$0 = \hat{b}\times(\nabla\times\vec{B}) = \nabla B - \hat{b}\cdot\nabla\underbrace{\vec{B}}_{\vec{B}=\hat{b}B}$$

allows us to relate $\vec{K}$ w/ grad B

$$0 = \nabla B - B\vec{K} - \underbrace{\hat{b}\hat{b}\cdot\nabla\vec{B}}_{\text{stuff parallel to } \vec{B}}$$

This directly impacts $\vec{v}_\perp$!

$$\vec{v}_{\perp 0} = \frac{c\vec{E}\times\vec{B}}{B^2} + \frac{c}{qB}\left(\mu\,\hat{b}\times\nabla\vec{B} + m v_{\parallel}^2\,\underbrace{\hat{b}\times\vec{\kappa}}_{\hat{b}\times\frac{1}{B}\nabla B}\right)$$

— parallel stuff goes away

curvature & grad B drift just kind of add together in the case of vacuum $\vec{B}$

$$\vec{v}_{\perp 0} = (\;) + \frac{c}{qB}\left(\mu + \frac{m v_{\parallel}^2}{B}\right)\hat{b}\times\nabla\vec{B}$$