*Britt's notes

Lecture 16 – Particle Orbits in Magnetic Fields 10/19/17

Particle Orbits in Magnetic Fields

{Ref: G+R Ch.3, Bellan Ch.3}

(1) Plasmas now have ambient $\vec{B}$-fields in them

↳ How to figure out whether particles are gaining energy or not (motion wrt $\vec{E}$-fields determines this)

Uniform Mag. Field (no $\vec{E}$-field)

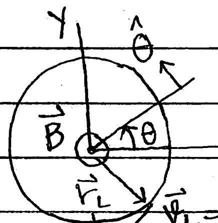
$$m\frac{d}{dt}\vec{v}_\perp = \frac{q}{c}\vec{v}_\perp \times \vec{B}, \quad m\frac{d}{dt}\underbrace{v_\parallel}_{\text{constant}} = 0$$

↓ dot w/ $\vec{v}_\perp$ ↓

$$\vec{v}_\perp \cdot \frac{d}{dt}\vec{v}_\perp = \frac{1}{2}\frac{d}{dt}v_\perp^2 = 0$$

$v_\perp$ = constant (magnitude)

⇒ corresponds to circular motion



$* \hat{\theta} = -\sin(\theta)\hat{x} + \cos(\theta)\hat{y}$

Look @ vector components of eq. of motion

Then in radial coords:

$$\underbrace{m v_\theta \frac{d}{dt}\hat{\theta}}_{\frac{d}{dt}\vec{v}_L} = \frac{q}{c}\vec{v}\times\vec{B} \qquad \frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{v_\theta}{r_L}\frac{d}{d\theta}$$

$$= \frac{q}{c} v_\theta \underbrace{\hat{\theta}\times\hat{z}}_{\text{in } \hat{r}\text{-direc}} B, \quad \hat{r} = \cos(\theta)\hat{x} + \sin(\theta)\hat{y}$$

$$= \frac{q}{c} v_\theta B \hat{r}$$

$\vec{v}_L$ = larmor velocity

$$m v_\theta v_\theta \frac{1}{r_L}\underbrace{\frac{\partial}{\partial\theta}\hat{\theta}}_{-\hat{r}} = \frac{q}{c} v_\theta B \hat{r}$$

$-\hat{r}$; direction of change in $\hat{\theta}$ w/ changing θ

$$\frac{-m v_\theta^{\cancel{2}}}{r} = \frac{q}{c}\cancel{v_\theta} B \qquad \text{gyro freq.} = \frac{qB}{mc}$$

$= -\frac{qB}{mc}$

$$\Rightarrow \begin{cases} v_\theta = -\Omega r_L \\ \vec{r}_L = \dfrac{-\vec{v}_L \times \vec{\Omega}}{\Omega^2} \end{cases}$$

$\vec{\Omega}$ – angular velocity

Circular motion eqns

$$= \frac{-v_\theta\Omega}{\Omega^2}(\hat{\theta}\times\hat{z})$$

→

NOW: Add E-field — time-uniform, spatially independent

Uniform $\vec{E}$ & $\vec{B}$ fields

$$m\frac{d}{dt}\vec{v} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$$

$$\rightarrow E_{\parallel} = \frac{\vec{E}\cdot\vec{B}}{B} = 0 \Rightarrow E_{\parallel} = 0$$

define — velocity we'd see in a frame moving at $\frac{c}{B^2}\vec{E}\times\vec{B}$

$$\vec{v} = \hat{\vec{v}} + \underbrace{\frac{c}{B^2}\vec{E}\times\vec{B}}_{\text{"E×B Drift"}}$$

non-relativistic velocity addition

time deriv of this $= 0$

$(\vec{v}_E)$ → no particle energy chng. → no current density

"E×B Drift"

└ in frame moving w/ this vel., NO E field

$$m\frac{d}{dt}\hat{\vec{v}} = q\vec{E} + \underbrace{\frac{q}{c}\frac{c}{B^2}(\vec{E}\times\vec{B})\times\vec{B}}_{-q\vec{E}} + \frac{q}{c}\hat{\vec{v}}\times\vec{B}$$

$A\times(B\times C) = (A\cdot C)B - (A\cdot B)C$; $(B\times C)\times A = -(A\cdot C)B + (A\cdot B)C$

$\Rightarrow (E\times B)\times B = -(B\cdot B)E + (B\cdot E)B = -B^2 E$

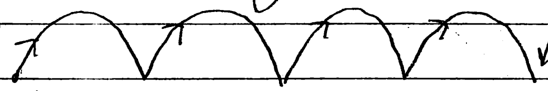
$$m\frac{d}{dt}\hat{\vec{v}} = \frac{q}{c}\hat{\vec{v}}\times\vec{B} \longrightarrow \vec{v}_E = \frac{c}{B^2}\vec{E}\times\vec{B} = \text{E-drift}$$

└ where here we've transformed to E-drift frame where there is NO electric field

Ex: Particle initially @ rest

$\vec{E}$ ↑ $\vec{B}$ ⊙

• starts in the direction of E, then as v increases, • moves under the influence of B, too



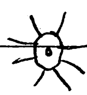
particle periodically stopped (or moving backward w/ v_L in v_E frame)

└ "cuspy orbit" — no chng. of total particle energy in time

In the v_E frame, particle is just gyrating around

└ only option for Larmor velocity is the drift velocity → $\frac{cE}{B}$ ⇒ Energy gained: $\frac{1}{2}m\left(\frac{cE}{B}\right)^2$

(no chng in KE in moving frame; no EPE here) — electric pot. energy



solar wind → E, B large

400–800 km/s

← 20 km/s — neutral particles - ionized in solar wind essentially @ rest

"pick-up particles" undergo cuspy orbit

→ gain Energy $\frac{1}{2}m\underbrace{\left(\frac{cE}{B}\right)^2}_{(400\ km/s)^2\text{ for protons}}$

Uniform B, $\vec{E}(t)$ — time-dep

$E_{\parallel} = 0$ still < starting w/ the the same approach we used when E was time independent

$m\frac{d}{dt}\vec{v} = q(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})$

$\hookrightarrow \vec{v} = \tilde{\vec{v}} + \underbrace{\frac{c}{B^2}\vec{E}\times\vec{B}}$

time deriv no longer 0 → must keep this

↓ plugging in ↓

$\underbrace{m\frac{d}{dt}\tilde{\vec{v}}} + m\frac{c}{B^2}\frac{d\vec{E}}{dt}\times\vec{B} = \underbrace{\frac{q}{c}\tilde{\vec{v}}\times\vec{B}}$

balancing these gives gyro motion

- consider a case where time-variation of $\vec{E}$ is slow compared to gyro time

$\hookrightarrow m\frac{c}{B^2}\frac{d\vec{E}}{dt}\times\vec{B}$ a small correction

define $\tilde{\vec{v}} = \tilde{\tilde{\vec{v}}} + \vec{v}_p$ — ~~tensor~~ velocity we'd see in a frame moving @ $\vec{v}_p$ (NOT a tensor)

solve for this

then

$m\frac{d}{dt}\tilde{\tilde{\vec{v}}} + \cancel{m\frac{d}{dt}\vec{v}_p} + \underbrace{\frac{mc}{B^2}\frac{d\vec{E}}{dt}\times\vec{B}} = \underbrace{\frac{q}{c}\vec{v}_p\times\vec{B}} + \frac{q}{c}\tilde{\tilde{\vec{v}}}\times\vec{B}$

throw out this; small comp. w/ ($\frac{d}{dt}\vec{v}_p$ small for slow time var.)

want to elim this by balancing w/ this

└ will define residual motion beyond gyro motion

we have (wtf?)

$\frac{mv_p}{\tau}, \frac{q}{c}v_pB, \frac{v_p}{\tau}, v_p\Omega$ — cyclotron freq.

(τ time scale (slow); charact. over which E is chg

→ cross w/ $\vec{B}$ to get impt. values

→

gets value for $\vec{v}_p$

$$\vec{B}\times\left(\frac{mc}{B^2}\frac{d\vec{E}}{dt}\times\vec{B} = \frac{q}{c}\vec{v}_p\times\vec{B}\right)$$

$$\hookrightarrow \frac{mc}{B^2}\underbrace{\vec{B}\times\left(\frac{d\vec{E}}{dt}\times\vec{B}\right)}_{\cancel{B^2}\frac{d}{dt}\vec{E}} = \frac{q}{c}\vec{v}_p \cancel{B^2}$$

Recall, BAC CAB expansion from 680

$$\frac{mc^2}{q B^2}\frac{d}{dt}\vec{E} = \frac{q}{\cancel{c}}\vec{v}_p \qquad \vec{v}_p = \frac{mc^2}{qB^2}\frac{d\vec{E}}{dt} = \frac{q}{m\Omega^2}\frac{d\vec{E}}{dt}$$

$$\frac{qB}{mc} = \Omega$$

$\vec{v}_E$ — $\vec{v}_E$ is $\perp$ to $\vec{E}$, so no energy change from ExB drift

$$\Rightarrow \vec{v}_p = \frac{1}{\Omega}\frac{d}{dt}\underbrace{\left(\frac{c\vec{E}}{B}\right)}_{\vec{v}_E}$$ Polarization Drift

units = velocity → like ExB drift, but in the direction of $\vec{E}$ ($\vec{v}_p$ → being in the direction of $\vec{E}$ means we have energy gain/loss)

$$\frac{v_p}{v_E} = \frac{1}{\Omega}\frac{d}{dt} \sim \frac{1}{\Omega\tau} \ll 1$$ ← this ~~shows~~ shows polarization drift is slow

Polarization Drift =
Energy gain associated w/ ExB drift
↳ if E chng w/ time, v_E chng w/ time and KE of part. also chng.
(no change in $\vec{E}$ → particles have uniform motion)

What does v_p correspond to?
It's the
Energy assoc. w/ v_E

$$W = \frac{1}{2}m v_E^2$$

$$\frac{dW}{dt} = \hookrightarrow \dot{W} = \frac{1}{2}m\frac{d}{dt}(\vec{v}_E)^2 = \frac{1}{\cancel{2}}m\cancel{2}\vec{v}_E\cdot\frac{d\vec{v}_E}{dt}$$

$$= m\frac{c}{B^2}\underbrace{\vec{E}\times\vec{B}}\cdot\frac{c}{B^2}\underbrace{\left(\frac{d\vec{E}}{dt}\times\vec{B}\right)}$$ flip positions

$$A\cdot(B\times C) = -B\cdot(A\times C)$$

$$= m\left(\frac{c}{B^2}\right)^2(\vec{E}\times\vec{B})\cdot\left(\frac{d\vec{E}}{dt}\times\vec{B}\right) = m\left(\frac{c}{B^2}\right)^2\left\{-\frac{d\vec{E}}{dt}\cdot\left[(\vec{E}\times\vec{B})\times\vec{B}\right]\right\}$$

$$= -\frac{mc^2}{B^4}\frac{d\vec{E}}{dt}\cdot\underbrace{\left[(\vec{E}\times\vec{B})\times\vec{B}\right]}_{-B^2\vec{E}}$$ (neg b/c of location of ~~singular~~ ×B)

$\dot{W} \frac{(q)mc^2}{(q)B^2} \vec{E} \cdot \frac{d\vec{E}}{dt}$

insert, now write wrt Ω

$\Rightarrow \frac{qc}{B} \frac{\vec{E}}{\Omega} \cdot \frac{d\vec{E}}{dt}$, $\vec{v}_p = \frac{1}{\Omega}\frac{d}{dt}\left(\frac{c\vec{E}}{B}\right)$

$\Rightarrow \dot{W} = q\,\vec{v}_p \cdot \vec{E}$

acquisition of energy due to polarization drift

*what exactly is char. time though? $\frac{r_L}{v_L}$?

NOW: move past the "Basics" and calc. orbits in more complicated systems

General Particle Drift in Space & Time Varying Fields

Assume ① slow time variation ($\Omega\tau \gg 1$) the characteristic time

τ is a long time wrt gyro freq)

② w/ $\frac{d}{dt}\vec{B} = 0$

spatial var. in B only

③ Weak space variation (comp. to gyro orbit scale)

- space scale L
- gyro scale r_L

$\Big\} \; r_L \ll L$

④ $|\vec{E}| \ll |\vec{B}|$

$\to v_E \sim \frac{cE}{B} \ll c$ — no relativistic corrections

small

Will have

Motion along B AND Drifts across B!

must describe both & distinguish between them

Do this w/ $\to$ separation of time scales

- fast gyro motion

vs

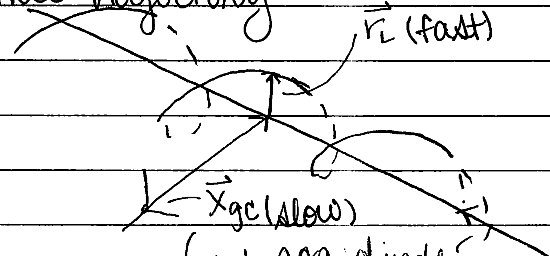
- slow drifts

Write position of particle as

$\vec{X}(t) = \vec{X}_{gc} + \vec{r}_L(t)$ — "Larmor-like" position

guiding center position

$\hookrightarrow \vec{v}(t) = \vec{v}_{gc} + \vec{v}_L(t)$



Two motion types: - rotational motion
- motion along center

- exp. EOM wrt. $\vec{x}(t)$ & $\vec{v}(t)$
assume $\vec{r}_L$ small - Tay exp.

$$\vec{B}(\vec{x}(t)) = \underbrace{\vec{B}(\vec{x}_{gc})}_{\vec{B}_{gc}} + \vec{r}_L \cdot \nabla \vec{B}(\vec{x}_{g.c}) + \ldots$$

$\vec{B}_{gc}$ mag field @ guiding center

∇B has scale $L \gg r_L$ — slow var

$$\vec{E}(\vec{x}(t)) = \underbrace{\vec{E}(\vec{x}_{gc})}_{\vec{E}_{gc}} + \vec{r}_L \cdot \nabla \vec{E}_{gc} + \ldots$$

Do the same for $\vec{E}$

$$m(\dot{\vec{v}}_{gc} + \dot{\vec{v}}_L) = q[\vec{E}_{gc} + \underbrace{\vec{r}_L \cdot \nabla \vec{E}_{gc}}] + \frac{q}{c}(\vec{v}_{gc} + \vec{v}_L) \times (\vec{B}_{gc} + \vec{r}_L \cdot \nabla B_{gc}) + \ldots$$

$\vec{E}_{gc}$: elec field @ guide center

$= 0$ when averaged over gyro motion ("avg. of spinning around just goes to 0")

Want to separate motion assoc. w/ fast gyro freq. v_L comp to that asso w/ slow guide ctr v_{gc}

$$m\dot{\vec{v}}_L = \frac{q}{c}\vec{v}_L \times \vec{B}_{gc}$$ → ~~subtract this from the eqn above and avg.~~ over gyro motion → linear terms in fast motion go away, remaining cross term is the product of two "fast" terms

~~[illegible]~~ ACTUALLY DO THIS →

$$\Rightarrow m\dot{\vec{v}}_{gc} = q\vec{E}_{gc} + \frac{q}{c}\vec{v}_{gc} \times \vec{B}_{gc}$$

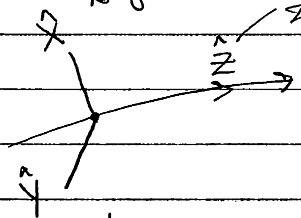
$$+ \frac{q}{c}\langle \vec{v}_L \times (\vec{r}_L \cdot \nabla)\vec{B}_{gc} \rangle$$

whole thing here is now slow time eqn.

$\llcorner$ time avg over gyro orbit / gyromotion (need to eval this)

→

local coord sys. based on local mag field



$\hat z$ along local B field

recall: $\vec r_L = -\frac{1}{\Omega}\vec v_L \times \hat z \;(= -\vec v_L\times\hat z/\Omega^2) = -\frac{1}{\Omega}(v_{Lx}\hat x + v_{Ly}\hat y)\times\hat z$

$$\hookrightarrow \begin{cases} r_{Lx} = -\frac{1}{\Omega} v_{Ly} \\ r_{Ly} = \frac{1}{\Omega} v_{Lx} \end{cases}$$

then

dot product here

$$\langle \vec v_L\times(\vec r_L\cdot\nabla)\vec B_{gc}\rangle = -\frac{1}{\Omega}\langle \vec v_L \times (\vec v_L\times\hat z)\cdot\nabla \vec B_{gc}\rangle = -\frac{1}{\Omega}\langle\vec v_L\times(\vec r_L\cdot\nabla)\vec B_{gc}\rangle$$

~~cross product of this~~

For the dot prod:
$\langle v_{Lx} v_{Ly}\rangle = 0$
sin & cos for circ. motion $\rightarrow \langle v_{Lx}^2\rangle = \frac{1}{2}v_L^2$

$$\cancel{= -\frac{1}{\Omega}\langle(v_{Lx}\hat x+v_{Ly}\hat y)\times([v_{Ly}\hat x - v_{Lx}\hat y]\cdot\nabla)\vec B_{gc}\rangle}$$

$$= -\frac{1}{\Omega}\langle (v_{Lx}\hat x+v_{Ly}\hat y)\times\left(v_{Ly}\frac{\partial}{\partial x} - v_{Ly}\frac{\partial}{\partial y}\right)\vec B_{gc}\rangle$$

$$= -\frac{1}{\Omega}\langle v_{Lx}v_{Ly}\frac{\partial}{\partial x}\hat x\times\vec B_{gc} - v_{Lx}^2\frac{\partial}{\partial y}\hat x\times\vec B_{gc} + v_{Ly}^2\frac{\partial}{\partial x}\hat y\times\vec B_{gc} - v_{Ly}v_{Lx}\frac{\partial}{\partial y}\hat y\times \vec B_{gc}\rangle$$

Surviving terms are then

$$= -\frac{1}{\Omega}\langle v_{Lx}\hat x\times\left(v_{Lx}\left(-\frac{\partial}{\partial y}\right)B_{gc}\hat z\right)\rangle$$

↓ rewrite wrt avgs ↓ $\;= -\frac{1}{\Omega}\frac{v_L^2}{2}\left(-\frac{\partial}{\partial y}\hat x\times\vec B_{gc} + \frac{\partial}{\partial x}\hat y\times\vec B_{gc}\right)$

$$= -\frac{v_L^2}{2\Omega}\underbrace{\left(\underbrace{\frac{\partial}{\partial x}\hat y\times\vec B}_{\text{acts on}} - \underbrace{\frac{\partial}{\partial y}\hat x\times\vec B}_{\text{acts on}}\right)}_{(\hat z\times\nabla)\times\vec B}$$

$\underbrace{(\hat z\times\nabla)\times\vec B}$ — ∇B_{gc}, grad guide ctr

$$= -\frac{v_L^2}{2\Omega}\nabla B_{gc}$$

Plugging this all back in for slow EOM

$$\cancel{m\dot{\vec v}_{gc} = q\vec E_{gc} + \frac{q}{c}\vec v_{gc}\times\vec B_{gc} - \frac{q}{c}\frac{v_L^2}{2qB}mc\,\nabla B_{gc}}$$

$\mu = \frac{\text{mag}}{\text{moment}} = \frac{m v_L^2}{2B}$ (an invariant)

* an adiabatic inv.

$$\Rightarrow m\dot{\vec v}_{gc} = q\vec E_{gc} + \underbrace{\frac{q}{c}\vec v_{gc}\times\vec B}_{\text{perp motion}} - \underbrace{\mu\nabla B_{gc}}_{\text{gives parallel forces}}, \text{ "Master eqn"}$$

defined as

*Mag mom:

$$\mu = \frac{IA}{c} = \underbrace{\frac{q}{\tau}}_{I}\,\underbrace{\frac{\pi r_L^2}{c}}_{A}$$

I, where $\tau = \frac{2\pi}{\Omega}$, $v_L = \Omega r_L$

plug in...

$$\cancel{\mu = \frac{q}{2\pi}\,\frac{\Omega^2}{\Omega}\,\frac{\pi r_L^2}{cqB}\,v_L^2}$$

$$\mu = \frac{q}{\tau}\,\frac{\pi r_L^2}{c}, \quad \tau = \frac{2\pi}{\Omega}$$

$v_L = r_L\Omega$

$$= q\,\frac{\Omega}{2\pi}\,\frac{\pi r_L^2}{c}\left(\frac{\Omega}{\Omega}\right) = \frac{q\,\Omega^2 r_L^2}{2c}\,\frac{1}{\Omega}$$

insert

$$= \frac{q v_L^2}{2c}\,\frac{1}{\Omega}, \quad \Omega = \frac{mc}{qB}$$

$$= \frac{q}{2c}\,\frac{mc}{qB}\,v_L^2 = \frac{m v_L^2}{2B}$$ ✓ as defined earlier