* Brittany's Notes  * G&R Ch. 25 Notes
* Drake's Notes

# Lecture 15 — Evolution of Quasilinear Systems   10/17/17

## Quasilinear Equations for Bump-on-Tail Instability

describe non-linear evolution of distr. f'n of particles in velocity space (weak turbulence — no strong trapping) from inter. w/ waves (neglecting wave-wave interaction)

$$\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v}\left\{D(v)\frac{\partial}{\partial v}f_0\right\} = 0 \quad \text{[Diff. Eq.]}$$

$$D(v) = \frac{e^2}{m^2}\sum_k \frac{k^2|\varphi_k|^2\gamma_k}{\gamma_k^2 + (\omega_k^R - kv)^2}$$

($\gamma_k$: growth rate of instability)

$\omega_k^2 = \omega_{pe}^2(1 + \text{therm corr})$ ← ignore

How dist. f'n evolves in time:

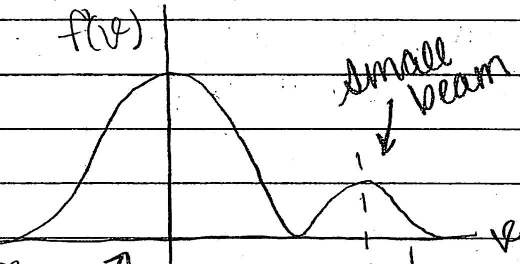
$$\frac{\partial}{\partial t}|\varphi_k|^2 = 2\gamma_k|\varphi_k|^2$$

sim?

$$\gamma_k = \frac{\pi}{2}\frac{\omega_{pe}^3}{k^2}\frac{1}{n_0}\left.\frac{\partial f_0}{\partial v}\right|_{\omega_k/k}$$

growth rate directly dependent on local slope of distr. function

→ number, momentum, and energy conserv. confirmed



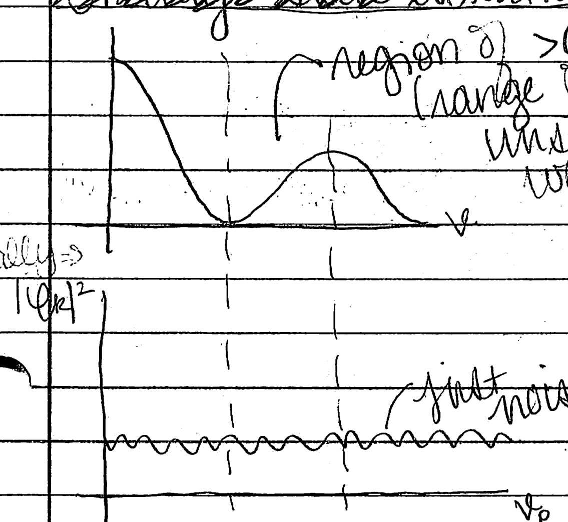

bulk of part. here (real freq. of wave doesn't chge, only $\omega_{pe}$ changes due to beam)

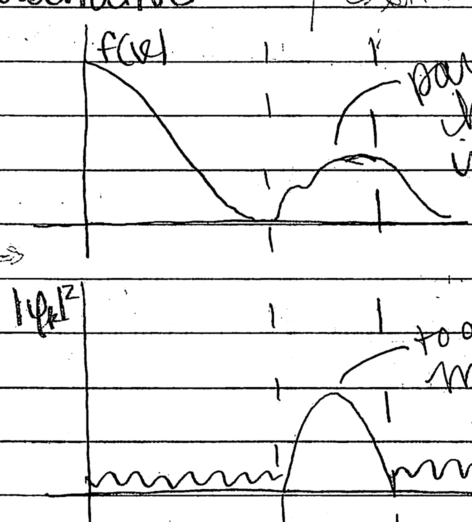
$\frac{\omega_k}{k} = \frac{\omega_{pe}}{k}$ phase speed

can vary $k$ to get desired phase speed of wave

source of free energy causes growth

## ~~Steady State~~ evolution of system — qualitative




Need to figure out how !! moves

Resonant particles are dragged to smaller velocities, giving energy to the wave

## Steady-state

No more evolution → $\partial f_0/\partial t = 0$

which implies

$$\frac{\partial}{\partial v}\left\{D\frac{\partial}{\partial v}f_0\right\}=0$$

↳ $D\frac{\partial}{\partial v}f_0 = \text{const.}$ (in $v$) $=0$ ← it's zero @ $\pm\infty$ (must be constant everywhere)

D must be finite ~~for~~ to still have energy in some region @ a late time ~~(long time)~~

⇒ $\frac{\partial f_0}{\partial v}$ must $=0$ if $D\neq 0$ (a plateau) "with energy added @ later times, we get:"

and

$\frac{\partial f_0}{\partial v}\neq 0$ if $D=0$

can be anything

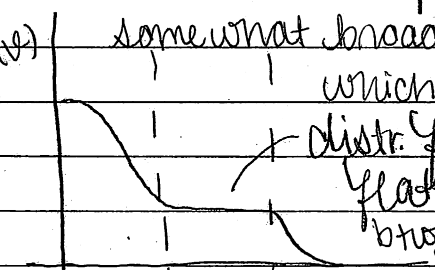
*During this evolution the spectrum spreads out somewhat, so as to extend throughout the range $k_{min}$ to $k_{max}$, corresponding to phase velocities from $v_1 \to v_2$, which is seen to be somewhat broader than the range of velocities for which the initial distribution has an inverted slope.

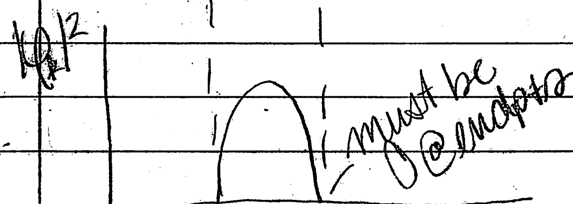
Translate into picture:

How do we get $v_1$ & $v_2$?

⇒ What are $v_1, v_2$?

must conserve # particles

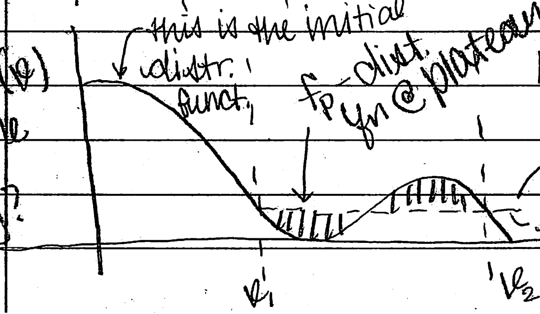



*Draw horizontal line on initial distribution when shaded areas are equal, number density is conserved and the intersections give $v_1$ & $v_2$

move this line (i.e., move the plateau) up & down until # part. in each region the same → that gives you $v_1$ & $v_2$ ($n=\int f\,dv$)

What are part. doing? (see above)



$v_2$ farthest pos. region (pg. 1) can move right

$f_0(v_1, t=0) = f_0(v_2, t=0)$ — distr. fn does not change (steady state)

$= f_0(v_1 < v < v_2,\ t=\infty)$ ← late time

$= f_p$

$$\boxed{\int_{v_1}^{v_2} dv\, f_0(v, t=0) = f_p(v_2 - v_1)}$$

↑ defines $v_1, v_2$

What about the wave spectrum?

energy Spectrum of Waves

- go back to time dev. —

Lorentzian — highly peaked around resonance ($\omega_k = kv$) for small $\gamma_k$

↳ treat as δ-fn

$$\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v}\frac{e^2}{m^2}\sum_k k^2 |\varphi_k|^2 \frac{\gamma_k}{\gamma_k^2 + \underbrace{(\omega_k^R - kv)^2}_{\text{resonant condition} \Rightarrow \text{largest when } 0}} \frac{\partial f_0}{\partial v} = 0 \quad \text{eq } ①$$

($\gamma_k$ small)

Where $\gamma_k(t)$, $|\varphi_k|^2(t)$, $f_0(t, v)$ — $\gamma_k$ taken to be small

How do we solve this?

↳ focus on resonant particles

- assumes weak growth (weak beam)

* $\frac{\partial f_0}{\partial v}$ related to $\gamma_k$ (from lec 14)

↳ $\frac{2}{\pi}\gamma_k \frac{k^2}{\omega_{pe}^3} n_0$ from

$$\lim_{\gamma_k \to 0} \frac{\gamma_k}{\gamma_k^2 + (\omega_k^R - k\cdot v)^2} = \pi\delta(\omega_k^R - k\cdot v)$$

$$\gamma_k = \frac{\pi}{2}\frac{\omega_{pe}^3}{k^2}\frac{1}{n_0}\left.\frac{\partial f_0}{\partial v}\right|_{v_p}$$

↓ plug in to eq. ①

would instead expand integral for $\omega_k \gg kv \gg \gamma_k$ in the non-resonant regions

$$\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v}\left\{\frac{e^2}{m^2}\sum_k k^2 |\varphi_k|^2 \pi\delta(\omega_k^R - kv)\frac{2}{\pi}\gamma_k \frac{k^2}{\omega_{pe}^3} n_0\right\} = 0$$

$\frac{\partial}{\partial t}|\varphi_k|^2 = 2\gamma_k|\varphi_k|^2$

combine $\frac{\partial}{\partial t}$:

$$\frac{\partial}{\partial t}\left[f_0 - \frac{\partial}{\partial v}\frac{e^2}{m^2} n_0 \sum_k \frac{k^4|\varphi_k|^2}{\omega_{pe}^3}\delta(\omega_k^R - kv)\right] = 0$$

↑ we've isolated the time-dependence

$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}_{}$

must be time indep. fun! — same @ final & initial times these must be equal to satisfy time-independence statement

no $\varphi$ here

$f_0 = f_p$ here

eval @ $t=0$ late time [on

$$f_0(v, t=0) = f_p - \frac{\partial}{\partial v}\frac{e^2}{m^2} n_0 \sum_k \frac{k^4 |\varphi_k|^2}{\omega_{pe}^3}\delta(\omega_k^R - k v)$$

must deal w/ this to get spectrum of k-values @ late time

*recall - only looking @ region w/ resonant particles (plateau only region of drastic change)

first isolate $\frac{\partial}{\partial v}$ and integrate

some arbitrary $v > v_1$

$$\frac{e^2}{m^2} n_0 \sum_k \frac{k^4|\varphi_k|^2}{\omega_{pe}^3}\delta(\omega_k^R - k v) = \int_{v_1}^{v} dv \left[f_p - f_0(v, t=0)\right] \quad \text{eq. (2)}$$

start of plateau

must conv. to integral

[periodic] Discrete Systems: length L

$k = \frac{2\pi n}{L}$, $\Delta k = \frac{2\pi}{L}$ $n$ = integer

$\to \frac{\Delta k L}{2\pi} = 1$

$$\Rightarrow \sum_k = \sum_k \left(\frac{\Delta k L}{2\pi}\right) = \underbrace{\frac{L}{2\pi}\int dk}_{\text{now a continuous system}}$$

$\sum_k \left(\frac{\Delta k L}{2\pi}\right) \to \int dk$

*Insert* $\sum_k k^4 |\varphi_k|^2\big|_{t=\infty}\delta(\omega_k - kv)$

$= \frac{L}{2\pi}\int dk(v) k^4 |\varphi_k|^2\big|_{t=\infty}\frac{\delta(\omega_k - kv)}{(v)}$ insert

$= \frac{L}{2\pi}\frac{k^2|E_k|^2|_\infty}{v}\Big|_{k = \omega_{pe}/v}$

NOW: $\int_{v_1}^{v} dv[f_p - f_0(v, t=0)] = \frac{e^2}{m^2} n_0 \frac{L}{2\pi}\int dk \frac{k^4|\varphi_k|^2}{\omega_{pe}^3}\delta(\omega_k^R - kv)$

*Int. over $\delta$-fn:

details see below

$$\int_{-\infty}^{\infty} dk\, \delta(g(k)) = \int_{-\infty}^{\infty}\frac{dk}{dg} dg\, \delta(g(k)) = \frac{1}{|dg/dk|}$$ being careful w/ your signs

~~***insert***~~

$\longrightarrow$ eq. (2) then becomes

$\delta$-fn: $kv - \omega_k^R = 0$

$kv = \omega_{pe} \to k = \frac{\omega_{pe}}{v} = \frac{\omega_k^R}{v}$

$|E_k|^2 = k^2|\varphi_k|^2$ where $\frac{1}{2}\sum_k |E_k|^2 \to \frac{2}{\varepsilon_0}\int_{-\infty}^{\infty}\mathcal{E}(k)\,dk$ "spectral energy density"

$$\frac{e^2}{m^2}\frac{n_0}{\omega_{pe}^3}\frac{k^4|\varphi_k|^2}{\left|\frac{d}{dk}(kv - \omega_k^R)\right|}\Bigg|_{\frac{\omega_{pe}}{v} = k} = \int_{v_1}^{v} dv\left[f_p - f(v, 0)\right]$$

gives $v_g$ - write as $V$ (therm effects small)

$= V - v_g$

$\delta(ax) = \frac{1}{|a|}\delta(x)$ & $\int dx\, \delta(x-a) f(x) = f(a)$ so $\int dx\, \delta(bx - a) f(x) = \frac{f(a/b)}{|b|}$

therefore $\delta(\omega_k^R - kv) = \frac{1}{|-v|}\delta(k - \omega_k^R/v)$

$$= \frac{e^2}{m^2} \frac{n_0}{\omega_{pe}^{3/2}} \frac{L}{2\pi} \cdot \frac{\omega_{pe}}{v} \cdot \frac{k|E_k|^2}{v} \Big|_{k=\omega_{pe}/v} = \frac{e^2 n_0}{m v^2} \frac{m}{4\pi n_0 e^2} \frac{L}{2\pi} k|E_k|^2 \Big|_{k=\frac{\omega_{pe}}{v}} \left(\frac{2}{2}\right) = \frac{2}{m v^2} \frac{kL}{2\pi} \frac{|E_k|^2}{8\pi} \Big|_{k=\frac{\omega_{pe}}{v}}$$

C*insert*

$|\psi_k|^2$ in terms of $\vec{E}$-field

How did we get here? ↳ insert

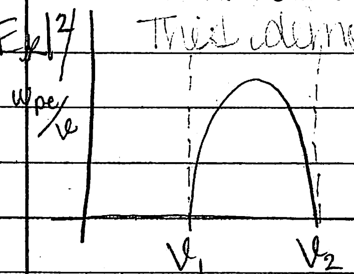
Rearranging...

$$\frac{kL}{2\pi} \frac{|E_k|^2_{t=\infty}}{8\pi} \Big|_{k=\frac{\omega_{pe}}{v}} = \frac{mv^2}{2} \underbrace{\int_{v_1}^{v}}_{} dv \left[ f_p - f_0(v,0) \right]$$

get 0 for $v_1$ & $v_2$ from $\int_{v_1}^{v_2} dv\, f_0(v, t=0)$ eqn

(late time wave energy spectrum wrt integral over distribution fn (valid for small growth rate)

~~This demonstrates that~~




$k_2 = \underbrace{\frac{\omega_{pe}}{v_2}}_{k_{min}}$

$\underbrace{k_1 = \frac{\omega_{pe}}{v_1}}_{k_{max}}$

$|E_k|^2\big|_\infty = 0$ for $k = \omega_{pe}/v_1$ and $\omega_{pe}/v_2$

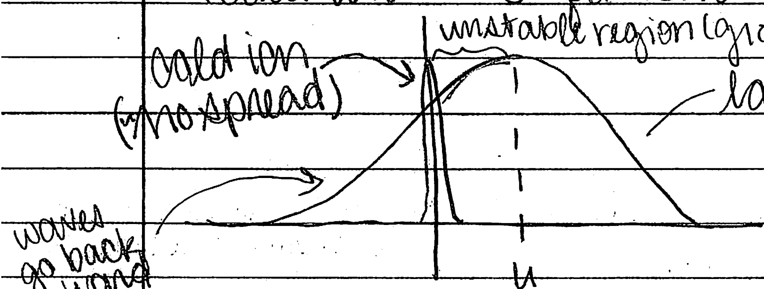
↳ at long time (spectral energy density falls to 0 at these points)

* waves exist btwn these two values of $k$, which gives their phase speed

## Comment on HW#6:

- ion acoustic waves

(cold ions → δ-function, essentially)




↳ drift velocity; $u \ll v_{te}$

ion acoustic dispersion relation

electrons are fast

$$\omega = \frac{kc_s}{(1 + k^2/k_D^2)^{1/2}}, \quad v_p = \frac{\omega_k}{k} \ll v_{te}, \quad 0 < v_p < c_s$$

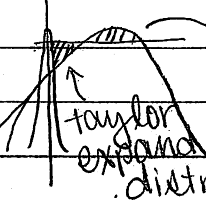
$k \to$ large ($v_1$)

$k = 0$ ($v_2$)

- if above this, no waves "up there" to flatten it

→ electrons will be diffusing (ignore ions – they're "dead cold")

⇒ becomes a plateau @ late time – all waves to the right –




plateau location

must conserve # of particles just w/ steady state plasma

distr. fn about 0 to find $v_2$ ($v_1 = 0$)

In exploring heating on diffusion of collisionless plasma, it is important to distinguish between reversible and non-reversible processes

NOW: Resonant vs non-resonant particles

Resonant vs. Non-Resonant Diffusion

→ must distinguish btwn reversible & irreversible processes ←

returning to diffusion eqn... consider non-resonant particles in bump-on tail case

diffusion coeff.

$$D(v) = \frac{e^2}{m^2}\sum_k \underbrace{\frac{k^2 |\varphi_k|^2 \gamma_k}{\gamma_k^2 + (\omega_k^R - kv)^2}}_{\text{approx. for res. particles by a } \delta\text{-fn}}$$

for non-res.

$$\Rightarrow \frac{e^2}{m^2}\sum_k \frac{(2)\,\gamma_k\, k^2\, |\varphi_k|^2}{(2)\,\omega_k^{R\,2}}$$

$\frac{\partial}{\partial t}|\varphi_k|^2$

insert

threw out $\delta$-fn

↓ write as time derivative ↓

$$= \frac{\partial}{\partial t}\left\{ \frac{e^2}{m^2}\sum_k \frac{\cancel{k^2|\varphi_k|^2} \rightarrow |E_k|^2}{2\,\omega_k^{R\,2}} \right\}$$

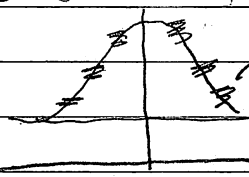
$$= \frac{\partial}{\partial t}\left\{ \frac{1}{2} |\hat{v}_k|^2 \right\}$$ ignoring $\sum_k$ (can put it in, too)

$e|E_k|$ = electric force magnitude

acceleration of $e^-$

$$\frac{e}{m}\frac{|E_k|}{\omega} = \hat{v}_k \text{ of } e^- \text{ in non-resonant space}$$

(jitter vel. of non-res part.



non-res just oscillate/jitter in place

time independent

$$\frac{\partial}{\partial t}\left( f_0 - \frac{\partial}{\partial v}\sum_k \frac{1}{2}|\hat{v}_k|^2 \frac{\partial f_0}{\partial v} \right) = 0$$

same form as "spectrum of waves" associated w/ the sloshing of nonresonant particles in unstable waves

"fake heating" of electrons (not REALLY heated)

b/c reversible (waves go away - jittering stops and

⇒ $f_0$ totally unchanged! That's it! "heating" goes away — this reverses as wave spectrum decays away

UNLIKE res. part. which are irreversibly changed

*Same for spatial diffusion! No irreversible diffusion in space of non-res. particles

☆TURNING POINT☆

previously left out

Beyond here - ambient magnetic fields present