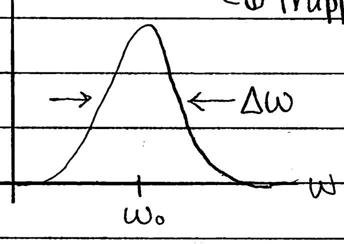
Lecture 14 – Quasilinear Theory 10/12/17

From last time...

Wave–Particle Interactions {G&R Ch. 25}

⇒ Two distinct limits depending on spectral width $\Delta\omega$

└ ① trapping; ② random interactions



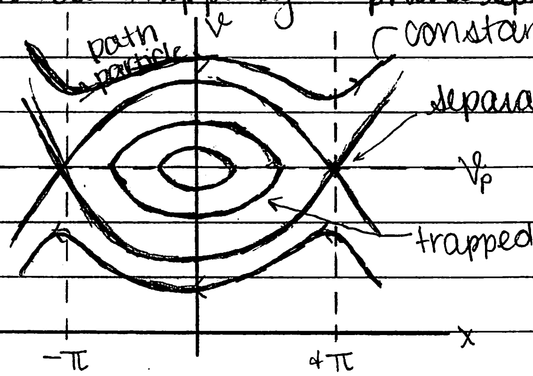
· Bounce frequency of deeply trapped particles in a stationary wave

$$\omega_B = \sqrt{\frac{k^2 e\varphi}{m}}$$

* the bounce timescale cannot be described by linear theory

① For $\omega_B \gg \Delta\omega$ (narrow wave spectrum) particles moving close to the phase speed of the wave see a quasi-DC field

└→ particle trapping



constant energy contours

separatrix (the boundary separating two modes of behavior in a differential equation)

critical points: critical energy beyond which the particle is no longer trapped

constant Energy Contours in v_j, x phase space

② For $\omega_B \ll \Delta\omega$ (broad wave spectrum) particles moving w/ very different velocities from v_p see an oscillatory field

└ the particle sees many waves and sees a coherent acceleration only for a time

→ no particle trapping

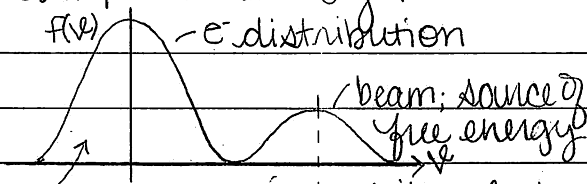
· instead get random scattering of particles off waves

⇒ Describe these using Quasilinear Theory

→

Quasilinear Treatment of Wave-Particle Interactions

→ Bump-on-Tail System ←



bulk of particles are here

i-drift velocity

→ beam causes growth/wave instability

(real freq. of wave doesn't change, only ω_{pe} changes due to beam)

Consider a plasma with a weak electron beam w/ $n_b \ll n_0$, where n_b is the beam density and n_0 is the background ion density

→ ⇒ The beam will drive the plasma waves

- ignore ion dynamics (take ions to be stationary) — we're looking @ high frequency plasma waves
- Work in 1-dimension

Starting with...

Vlasov-Poisson System of Equations (1D)

• Vlasov:

$$\underbrace{\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \varphi}{\partial x}}_{= F/m = \dot{v} = \frac{-e}{m}E = \frac{e}{m}\partial\varphi/\partial x}\frac{\partial f}{\partial v} = 0$$

$E = -\partial\varphi/\partial x$

• Poisson:

$$\nabla\cdot E = 4\pi\rho \rightarrow \frac{\partial}{\partial x}E = 4\pi e(n_0 - \underbrace{n_e}_{n_e = \int dv\, f})$$

(background density)

$$\underbrace{\frac{\partial}{\partial x}E}_{= -\partial^2\varphi/\partial x^2} = 4\pi e n_0\left(1 - \frac{1}{n_0}\int dv\, f\right)$$

$$\frac{\partial^2\varphi}{\partial x^2} = -4\pi e n_0\left(1 - \frac{1}{n_0}\int dv\, f\right)$$

→

Linear Theory

$\varphi \sim Re\{\underset{\text{complex amplitude}}{\varphi_k} e^{ikx} e^{-i\omega t}\}$

$f = \underbrace{f_0(v,t)}_{\text{allow } f_0(v,t) \text{ to vary in time slowly}} + Re\{f_k e^{ikx} e^{-i\omega t})$ ← background distribution; space independent

Start by linearizing the Vlasov-Poisson system, assuming that amplitude $\times e^{ikx}$ is small:

$$\left.\begin{aligned}(-i\omega + ikv) f_k + \frac{e}{m} ik\varphi_k \frac{\partial f_0}{\partial v} = 0 \\ k^2 \varphi_k = 4\pi e\left(-\int dv\, f_k\right)\end{aligned}\right\} \text{combine these}$$

$$(-i\omega + ikv) f_k = \frac{-e}{m} ik\varphi_k \frac{\partial f_0}{\partial v}$$

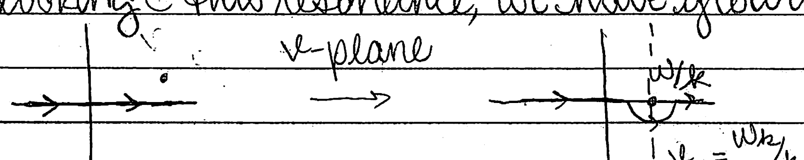
$$f_k = \frac{-e/m\, k\varphi_k\, \partial f_0/\partial v}{(-\omega + kv)}$$

$$k^2\varphi_k = 4\pi e\left[+\int dv \frac{+e/m\, k\varphi_k\, \partial f_0/\partial v}{(-\omega + kv)}\right]\underbrace{\left(\frac{n_0}{n_0}\right)}_{\text{insert}}$$

$$1 = \int dv \underbrace{\frac{4\pi e^2 n_0}{m}}_{\omega_{pe}^2} \frac{1}{kn_0} \frac{\partial f_0/\partial v}{\underbrace{(-\omega + kv)}_{= -(\omega - kv)}}$$

$$\Rightarrow 1 + \frac{\omega_{pe}^2}{k}\frac{1}{n_0}\int dv \underbrace{\frac{1}{\omega - kv}}_{\text{resonance}} \frac{\partial f_0}{\partial v} = 0 \quad \text{Dispersion Relation}$$

Looking @ this resonance, we have growing modes for $Im\{\omega\} > 0$



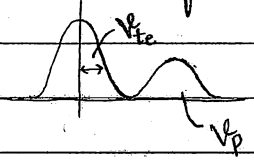
- Take $Im\{\omega\} > 0$, but small
  - ↳ break up the above integral

→

$$1 - \frac{\omega_{pe}^2}{k^2 n_0} \underbrace{P\int dv \frac{\partial f_0}{\partial v} \frac{1}{v - \omega/k}}_{\text{this will have an imaginary part due to } \omega} - \frac{\omega_{pe}^2}{k^2 n_0} i\pi \frac{\partial f_0}{\partial v}\bigg|_{v=\omega/k} = 0$$

principal value integral

Take $v_p \gg v_{te}$ for background



$\Rightarrow \omega/k \gg v_{te}$

↓ integrate by parts ↓

$$1 - \frac{\omega_{pe}^2}{k^2 n_0} \underbrace{P\int dv \frac{f_0}{(v - \omega/k)^2}}_{\simeq k^2/\omega^2 + \text{thermal corrections}} - \frac{\omega_{pe}^2}{k^2 n_0} i\pi \frac{\partial f_0}{\partial v}\bigg|_{v=\omega/k} = 0$$

since these are small, as shown above

thermal corrections ... which one?

$$1 - \frac{\omega_{pe}^2}{\omega^2}\left(\frac{1}{n_0}\right)\left[1 + (\ \ )\right] - i\pi \frac{\omega_{pe}^2}{k^2 n_0} \frac{\partial f_0}{\partial v}\bigg|_{v=\omega/k} = 0 \qquad 3k^2 \frac{T_e}{m_e \omega^2} = \frac{3}{2}\frac{k^2 v_{th}^2}{\omega^2}$$

* Thermal corrections are such that

$$\omega_k^2 = \omega_{pe}^2 + \frac{3T_e k^2}{m_e}$$

* ~~this seems wrong~~ → ~~lec #10, pg. 3 top~~

Let $\omega = \omega_{pe} + \Delta\omega$ → do a Taylor series expansion

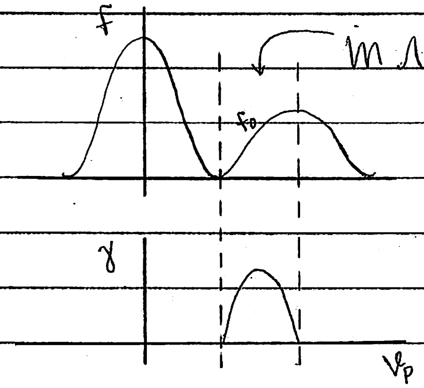
$$1 - \frac{\omega_{pe}^2}{(\omega_{pe} + \Delta\omega)^2}\left[1 + (\ )\right] - i\pi \frac{\omega_{pe}^2}{k^2 n_0} \frac{\partial f_0}{\partial v}\bigg|_{v=\omega/k} = 0$$

$$\frac{2\overbrace{\Delta\omega}^{i\gamma_k}}{\omega_{pe}} - i\pi \frac{\omega_{pe}^2}{k^2 n_0} \frac{\partial f_0}{\partial v}\bigg|_{v=\omega_k/k} = 0$$

~~what is happening here?~~

$$2\frac{\omega_{pe}^2}{\omega^3} i\gamma_k = i\pi \frac{\omega_{pe}^2}{k^2 n_0} \frac{\partial f_0}{\partial v}\bigg|_{v=\omega_k/k}$$

$$\Rightarrow \gamma_k = \frac{\pi}{2} \frac{\omega_{pe}^3}{k^2} \frac{1}{n_0} \underbrace{\frac{\partial f_0}{\partial v}\bigg|_{v=\omega_k/k}}_{\text{when this is positive, we have growing waves}}$$



in regions of positive slope ($\partial f_0/\partial v|_{v=\omega_{pe}/k} > 0$), we have an "unstable branch", yielding positive growth rate

Now, actually do

Quasilinear Theory

Assume that the dominant nonlinearity process occurs due to the resonant interaction of the waves with the particles

$\Rightarrow$ Find an equation for the uniform part of f, $f_0(v,t)$

Considerations:

- neglect wave-wave interactions
- plasma is 1D, uniform, and unmagnetized
- plasma characterized by coupled Vlasov & Poisson eqns of e^-
  - only electrostatic modes; ion motion neglected
  - ions are static, uniform neutralizing background
- electron velocity distribution function has two parts
  1. spatially independent term
     - slow temporal variation; appx. stationary over characteristic length
  2. small, high frequency perturbations that are space-time dependent
     - has spectrum of linear plasma waves

Generally f_p = plasma disp fn?

$$\frac{\partial f_p}{\partial t} + ipvf_p + \frac{e}{m} ip\varphi_p \frac{\partial f_0}{\partial v} + \frac{e}{m} \underbrace{ik\varphi_k}_{\text{summing over } k \text{ here}} \frac{\partial}{\partial v}(f_{p-k}) = 0$$ Collisionless Boltzmann

~~?where did we get all this?~~

→ take $p=0$ (this eliminates wave-wave interaction terms)

$$\frac{\partial f_0}{\partial t} + \frac{e}{m}\sum_k ik\varphi_k \frac{\partial}{\partial v}\underbrace{f_{-k}}=0$$

Recall: $f_k = \frac{e}{m} k\varphi_k \frac{\partial f_0}{\partial v}\frac{1}{\omega_k - kv}$

↓plugging in↓

$$\frac{\partial f_0}{\partial t} + \frac{e}{m}\sum_k ik\varphi_k \frac{\partial}{\partial v}\left[\frac{e}{m}(-k)\varphi_{-k}\frac{\partial f_0}{\partial v}\frac{1}{\omega_{-k}+kv}\right]=0$$

★ └why no sum over $-k$?

$$=\frac{\partial f_0}{\partial t} + \frac{e}{m}\sum_k ik\varphi_k \frac{\partial}{\partial v}\left(\frac{e}{m}\varphi_{-k}\underbrace{\frac{-k}{\omega_{-k}+kv}}_{\text{Reality conditions}}\right)\frac{\partial f_0}{\partial v}=0 \qquad \text{eq. ①}$$

↳ $\varphi_{-k} = \varphi_k^*$

↳ $\omega_{-k} = -\omega_k^* = -\omega_R(k) + i\gamma_k$ — Proof in Lec. #10

where $\varphi_{-k} = \varphi_k^*$ satisfies the requirement that

$$\varphi(\vec{x},t) = \varphi_k e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + \varphi_{-k}e^{-i\vec{k}\cdot\vec{x} - i\omega_{-k}t} = \text{REAL}$$

then eq. ① becomes

$$\frac{\partial f_0}{\partial t} + \frac{e}{m}\sum_k ik\varphi_k \frac{\partial}{\partial v}\left[\frac{e}{m}\varphi_k^*\frac{-k}{(-\omega_k^R + i\gamma_k)+kv}\right]\frac{\partial f_0}{\partial v}=0$$

$\varphi_k\varphi_{-k} = |\varphi_k|^2$

$$=\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v}\left(\frac{e^2}{m^2}\sum_k \frac{ik^2|\varphi_k|^2}{-\omega_k^R + i\gamma + kv}\frac{\partial f_0}{\partial v}\right)$$

$$=\frac{\partial f_0}{\partial t} \quad \frac{e^2}{m^2}\sum_k ik^2|\varphi_k|^2 \frac{\partial}{\partial v}\frac{\partial f_0/\partial v}{(-\omega_k^R + i\gamma)+kv}=0$$

★ * We can rewrite this as a classic Diffusion Equation (in velocity space) └Diffusion of particles?

$$\Rightarrow \frac{\partial f_0}{\partial t} - \frac{\partial}{\partial v}\underbrace{D(v)}_{\text{Diffusion coefficient}}\frac{\partial f_0}{\partial v}=0$$

$$D(v) = \frac{e^2}{m}\sum_k ik^2|\varphi_k|^2\frac{1}{(-\omega_k^R + i\gamma)+kv} \quad \text{from above}$$

$$D(v) = \frac{e^2}{m^2}\sum_k \frac{ik^2|\varphi_k|^2}{(-\omega_k^R + i\gamma_k + kv)}\left[\frac{-\omega_k^R - i\gamma_k + kv}{-\omega_k^R - i\gamma_k + kv}\right]$$

└ multiply by complex conjugate to get fully-real denominator

$$= \frac{e^2}{m^2}\sum_k \frac{|\varphi_k|^2 ik^2 \overbrace{(-\omega_k^R - i\gamma_k + kv)}}{(\omega_k^R - kv)^2 + \gamma_k^2}$$

$\sum_k(-\omega_k^R - i\gamma_k + kv) \to \sum_k(-i\gamma_k)$ b/c $(-\omega_k^R + kv)$ odd in k and thus is canceled out by $\sum_k$

~~$\sum_k(-\omega_k^R + kv)$ "cancels during sum $k,-k$" but we need reality conditions and therefore have no sum over $-k$...? Doesn't it just cancel over sum k b/c $(-\omega_k^R + kv)$ is odd in k?~~

$$= \frac{e^2}{m^2}\sum_k \frac{|\varphi_k|^2 ik^2(-i\gamma_k)}{(\omega_k^R - kv)^2 + \gamma_k^2}$$

$$\Rightarrow D(v) = \frac{e^2}{m^2}\sum_k \frac{k^2\gamma_k|\varphi_k|^2}{(\omega_k^R - kv)^2 + \gamma_k^2},$$

A complete set of quasilinear equations!

where

$$\gamma_k = \frac{\pi}{2}\frac{\omega_{pe}^3}{k^2}\frac{1}{n_0}\frac{\partial f_0}{\partial v}\bigg|_{v=\omega_k/k};$$

$$\frac{\partial}{\partial t}|\varphi_k|^2 = 2\gamma_k|\varphi_k|^2$$

what's the benefit of writing $D(v)$ wrt $\frac{\partial}{\partial t}$ of $|\varphi_k|^2$?

└ recall: $k^2|\varphi_k|^2 = \underbrace{e^{2\gamma_k t}}|E_k|^2$

sometimes written as an intrinsic part of E_k

Describes how the distr. function evolves in time

$$\to D(v) = \frac{e^2}{2m^2}\sum_k \frac{k^2}{(\omega_k^R - kv)^2 + \gamma_k^2}\frac{\partial}{\partial t}|\varphi_k|^2$$

Is it so you can collect the $\frac{\partial}{\partial t}$ terms with the $\partial f_0/\partial t$ in $\partial f_0/\partial t - \frac{\partial}{\partial v}D(v)\frac{\partial}{\partial v}f_0 = 0$?

Quasilinear Theory has nice conservation laws...

Conservation Laws

Conservation of Particles — Number Density

$$n = \int dv\, f_0$$

→ Go back to diffusion equation & integrate over velocity?
(the 0^{th} moment)

→

no space dependence

$$\frac{\partial}{\partial t}\Big\{\underbrace{\int dv\, f_0}_{n}\Big\} = \int_{-\infty}^{\infty} dv \frac{\partial}{\partial v}\left[D(v)\frac{\partial f_0}{\partial v}\right]$$

$$\frac{\partial n}{\partial t} = \left(D\frac{\partial f_0}{\partial v}\right)\Big|_{-\infty}^{+\infty}$$

goes to zero @ $\pm\infty$

$\partial n/\partial t = 0 \Rightarrow$ Total # density is a constant/ is conserved

Conservation of Momentum – Momentum Density

– Multiply the diffusion equation by v (now the 1st moment), then integrate this differential eq. wrt velocity

$$\frac{\partial}{\partial t}\left\{\int_{-\infty}^{\infty} dv\, v f_0\right\} = \int_{-\infty}^{\infty} dv\, v \frac{\partial}{\partial v}\left[D(v)\frac{\partial f_0}{\partial v}\right]$$

↓ integrate by parts ↓

$$= \underbrace{\left(vD\frac{\partial f_0}{\partial v}\right)\Big|_{-\infty}^{+\infty}}_{=0\text{, as above}} - \int_{-\infty}^{\infty} dv\left[D(v)\frac{\partial f_0}{\partial v}\right]$$

plug in 2

$$k^2|\varphi_k|^2 = e^{2\gamma_k t}|E_k|^2$$

$$= -\int_{-\infty}^{\infty} dv\left[\frac{e^2}{m}\sum_k \frac{k^2|\varphi_k|^2\gamma_k}{(\omega_k^R - kv)^2 + \gamma_k^2}\right]\frac{\partial f_0}{\partial v}$$

$$= -\int_{-\infty}^{\infty} dv \frac{e^2}{m^2}\sum_k |E_k|^2 e^{2\gamma_k t}\underbrace{\frac{\gamma_k}{(\omega_k^R - kv)^2 + \gamma_k^2}}\frac{\partial f_0}{\partial v}$$

$$\frac{\gamma_k}{\gamma_k^2 + (\omega_k - kv)^2} = -\frac{\mathrm{Im}\{\omega^*\}}{(\omega - kv)(\omega^* - kv)}$$

$$= -\mathrm{Im}\{1/(\omega - kv)\}$$

$$= +\int_{-\infty}^{\infty} dv \frac{e^2}{m^2}\sum_k |E_k|^2 e^{2\gamma_k t}\left[+\mathrm{Im}\left\{\frac{1}{\omega - kv}\right\}\right]\frac{\partial f_0}{\partial v}$$

☆ But we know — origin of this form of ε? — plug in for $\omega_{pe}^2 = ?$

$$\varepsilon = 0 = 1 + \frac{4\pi q^2}{mk^2}\left(\frac{n_0}{n_0}\right) = 1 + \frac{\omega_{pe}^2}{n_0 k}\int dv \frac{\partial f_0}{\partial v}\frac{1}{\omega - kv}$$

insert

dispersion relation

as discussed in Lec. #13, $\varepsilon = 0$ when looking for the normal modes of the system (thermal corrections negligible)

$$= \int_{-\infty}^{\infty} dv \frac{e^2}{m^2} \sum_k |E_k|^2 e^{2\gamma_k t} \, \text{Im}\left\{\frac{1}{\omega - kv}\right\} \frac{\partial f_0}{\partial v}$$

$$-1 = \frac{\omega_{pe}^2}{n_0 k} \int dv \frac{\partial f_0}{\partial v} \frac{1}{\omega - kv} \rightarrow \int dv \frac{\partial f_0}{\partial v} \frac{1}{\omega - kv} = \frac{-n_0 k}{\omega_{pe}^2}$$

$$\frac{\partial}{\partial t}\Big[\underbrace{\int_{-\infty}^{\infty} dv\, v f_0}_{\text{momentum, } p}\Big] = \frac{e^2}{m^2} \sum_k |E_k|^2 e^{2\gamma_k t} \underbrace{\text{Im}\left\{\frac{-n_0 k}{\omega_{pe}^2}\right\}}_{=0 \text{ since this is odd in } k \text{ and is real}}$$

~~was this introduced?~~ previously written as $\int d\vec{v} f_\alpha \vec{v} = n_\alpha \vec{u}_\alpha$

$\frac{\partial}{\partial t} p = 0 \Rightarrow$ momentum is a constant / is conserved

Conservation of Energy – Energy Density

- Mean particle kinetic energy: $W_p = \int dv \frac{mv^2}{2} f_0$
  - ↳ $\frac{\partial}{\partial t} W_p$ can be found by taking the 2nd moment, i.e., multiply the diffusion equation by $\frac{1}{2}mv^2$ and then integrate this with respect to the velocity

$$\frac{\partial}{\partial t}\left\{\int dv \frac{1}{2} m v^2 f_0\right\} = \frac{\partial W_p}{\partial t} = \int dv \frac{1}{2} m v^2 \frac{\partial}{\partial v}\left[D(v) \frac{\partial f_0}{\partial v}\right]$$

↓ integrate by parts ↓

$$= \underbrace{\left(\frac{1}{2} m v^2 D(v) \frac{\partial f_0}{\partial v}\right)\Big|_{-\infty}^{\infty}}_{=0, \text{ as before from behavior of } \partial f_0/\partial v \,@\, \pm\infty} - \int_{-\infty}^{\infty} dv \frac{1}{\cancel{2}} m(\cancel{2}v) \left[D(v) \frac{\partial f_0}{\partial v}\right]$$

plug in

$$= -\int_{-\infty}^{\infty} dv\, mv \left[\frac{e^2}{m^2} \sum_k \frac{\overbrace{k^2 |\varphi_k|^2}^{k^2|\varphi_k|^2 = e^{2\gamma_k t}|E_k|^2} \gamma_k}{(\omega_k^R - kv)^2 + \gamma_k^2}\right] \frac{\partial f_0}{\partial v}$$

$$= -\int_{-\infty}^{\infty} dv \frac{e^2}{m} \sum_k |E_k|^2 e^{2\gamma_k t} \underbrace{\frac{v \gamma_k}{(\omega_k^R - kv)^2 + \gamma_k^2}}\frac{\partial f_0}{\partial v}$$

$$\frac{\gamma_k}{\gamma_k^2 + (\omega_k^R - kv)^2} = + \text{Im}\left\{\frac{1}{(\omega - kv)}\right\}$$

$$= -\int_{-\infty}^{\infty} dv \frac{e^2}{m} \sum_k |E_k|^2 e^{2\gamma_k t} v \left[+ \text{Im}\left\{\frac{1}{(\omega - kv)}\right\}\right] \frac{\partial f_0}{\partial v}$$

same approach as in Lec. 9

$$\frac{\partial}{\partial t} W_p = \frac{-e^2}{m} \sum_k \frac{|E_k|^2}{k} e^{2\gamma_k t} \int dv \, \mathrm{Im}\left\{ \frac{kv + \omega - \omega}{\omega - kv} \right\} \frac{\partial f_0}{\partial v}$$

insert

$$= \frac{-e^2}{m} \sum_k \frac{|E_k|^2}{k} e^{2\gamma_k t} \int dv \, \mathrm{Im}\left\{ \overset{\text{zero}}{-1} + \frac{\omega}{\omega - kv} \right\} \frac{\partial f_0}{\partial v}$$

$$= \frac{-e^2}{m} \sum_k \frac{|E_k|^2}{k} e^{2\gamma_k t} \, \mathrm{Im}\left\{ \omega \int dv \frac{1}{\omega - kv} \frac{\partial f_0}{\partial v} \right\}$$

insert dispersion relation found for momentum conservation

$$= \frac{+e^2}{m} \sum_k \frac{|E_k|^2}{\cancel{k}} e^{2\gamma_k t} \, \mathrm{Im}\left\{ \omega \left(\frac{+n_0 \cancel{k}}{\omega_{pe}^2} \right) \right\}$$

$$= \frac{n_0 e^2}{\omega_{pe}^2 m} \underbrace{\sum_k |E_k|^2 e^{2\gamma_k t} \gamma_k}$$

where

$$|E_k|^2 e^{2\gamma_k t} \gamma_k = \frac{1}{2} \frac{\partial}{\partial t} \left(|E_k|^2 e^{2\gamma_k t} \right)$$

$$\frac{\gamma_k}{\gamma_k^2 + (\omega_R^2 - kv)^2} = \mathrm{Im}\left\{ \frac{1}{\underbrace{\omega - kv}_{\text{just } \omega \text{ above}}} \right\}$$

$$= \frac{n_0 e^2}{\omega_{pe}^2 m} \frac{1}{2} \sum_k \frac{\partial}{\partial t} \left(|E_k|^2 e^{2\gamma_k t} \right)$$

plug in for $\omega_{pe}^2 = 4\pi e^2 n_0 / m$

$$= \frac{\cancel{m} \cancel{n_0} \cancel{e^2}}{4\pi \cancel{e^2} \cancel{n_0} \cancel{m}} \frac{1}{2} \sum_k \frac{\partial}{\partial t} \left(|E_k|^2 e^{2\gamma_k t} \right)$$

$$= \frac{\partial}{\partial t} \left(\sum_k \frac{1}{8\pi} |E_k|^2 e^{2\gamma_k t} \right)$$

⇒ Then, putting this together

$$\frac{\partial}{\partial t} \Big(\underset{\text{KE}}{W_p} + \underbrace{\sum_k \frac{1}{8\pi} |E_k|^2 e^{2\gamma_k t}}_{\text{field energy}} \Big) = 0$$

Total energy is conserved!