Lecture 13 - Fluctuations & Interactions   10/10/17

*Goal of HW#5, Problem 1:

→ demonstrate reality condition $\omega_{-k} = -\omega_k^*$

where $\omega_k = \omega_k^R + i\underbrace{\gamma_k}_{}$

$\gamma_k < 0$ corresponds to a damped wave

Then, $\omega_{-k} = \omega_{-k}^R + i\gamma_{-k}$

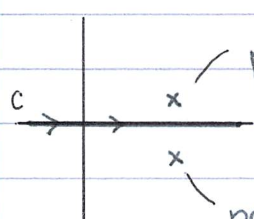
└ will find $-\omega_k^R$

We must work out an expression for $-k$!

- Cannot be $\omega_{-k} = -\omega_k$ because in this case, a damped mode in k would be a growing mode in $-k$

⇒ Must have same mode type in $\pm k$

Evaluate $\int d\vec{v} \frac{1}{\vec{v} - \omega/\vec{k}} \frac{\partial f_0}{\partial \vec{v}}$ where in performing the Lorentz transform, we require $\underbrace{Im\{\omega\}}_{\text{gives direction in time}} > 0$ (growth)



pole here for $+k$

↳ deform C down

pole here for $-k$

↳ deform C up (changes sign of residue)

Note: In general, just never cross the pole with your contour, regardless of location in UHP or LHP

⇒ Switching $k \to -k$ should yield the same damping rate

from last time...

Energy Density & Energy Flux in General EM Waves

• Energy density:

$$W = \frac{\partial}{\partial \omega_0} \frac{(\omega_0 \vec{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \vec{E}_0)}{8\pi}$$

$\overline{\overline{G}}_0 = \overline{\overline{G}}(\omega_0, \underbrace{\vec{k}_0}_{})$

Expansion about a specific k; the dominant wave vector

*Recall: the momentum of the wave is directly related to the

KLE

energy content of the wave

- Energy flux:

$$\vec{\Gamma} = -\frac{\partial}{\partial \vec{k}_0}\left(\frac{\omega_0 \vec{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \vec{E}_0}{8\pi}\right)$$

centered around dominant wave vector w/ some spread

$$= W \underbrace{\vec{v}_g}$$

Group Velocity — related to energy spread

$$\vec{v}_g = \frac{d\omega_{k_0}}{d\vec{k}_0} < c \text{ always}$$

NOW: Starting with the general dispersion relation

$$\overline{\overline{G}} = \frac{c^2}{\omega^2}\left(\vec{k}\vec{k} - \overline{\overline{I}}k^2\right) + \overline{\overline{\varepsilon}}$$

where

$$\overline{\overline{\varepsilon}} = \varepsilon_\parallel \frac{\vec{k}\vec{k}}{k^2} + \varepsilon_\perp\left(\overline{\overline{I}} - \frac{\vec{k}\vec{k}}{k^2}\right), \text{ the dielectric tensor}$$

$$\varepsilon_\perp = 1 + \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{kv_{te}} Z\underbrace{\left(\frac{\omega}{kv_{te}}\right)}_{\zeta_e}$$

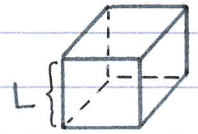
$$\varepsilon_\parallel = 1 - \frac{k_{De}^2}{2k^2} Z'\left(\frac{\omega}{kv_{te}}\right)$$

For particles streaming around, we do not expect the plasma to be fully quiescent

⇒ we expect thermal fluctuations/waves!

Thermal Fluctuations

We have a system of characteristic scale length L



→ L allows for a discrete set of modes, i.e., makes the spectrum of wave vectors $\vec{k}$ in the system discrete

$$\Delta k = \frac{2\pi}{L}, \quad d^3k = \left(\frac{2\pi}{L}\right)^3$$

→

Beginning with Poisson's equation...

$\nabla \cdot \vec{E} = 4\pi\rho$

$= 4\pi e(n_e - n_i)$

└ not generalized for all Z-values

Perform a discrete transformation

(want to evaluate over all k-values)

$$\vec{E}_k = \int \frac{d^3x}{L^3} e^{-i\vec{k}\cdot\vec{x}} \underbrace{\vec{E}(\vec{x})}_{\vec{E}(\vec{x}) = \sum_j e^{i\vec{k}_j\cdot\vec{x}} \vec{E}_{k_j}}$$

$$= \underbrace{\sum_j e^{i\vec{k}_j\cdot\vec{x}} \int \frac{d^3x'}{L^3} e^{-i\vec{k}_j\cdot\vec{x}'}}_{= \int d^3x' \delta(\vec{x}-\vec{x}')\text{; similar to previous evaluations over } k} \vec{E}(\vec{x}')$$

$$= \int d^3x' \delta(\vec{x}-\vec{x}') \vec{E}(\vec{x}')$$

$\vec{E}_k = \vec{E}(\vec{x})$

↳ E_k is intrinsically space-dependent

Consider the plasma response to a discrete particle moving with some velocity → a "dressed particle"

(Refers to a bare particle (an excitation of an elementary quantum field) together with some excitations of other quantum fields that are physically inseparable from the bare particle

- Starting with Poisson's equation

$\nabla \cdot \vec{E} = 4\pi\rho \rightarrow i\vec{k}\hat{E} = 4\pi\hat{\rho}$

└ $\hat{E} = -i\vec{k}\hat{\varphi}$

$k^2\hat{\varphi} = 4\pi\hat{\rho}$

PREVIOUSLY: When looking for normal modes; Ref: Lec. #8

$k^2\hat{\varphi} - 4\pi\hat{\rho} = 0$

$$k^2 \underbrace{\left(1 - \frac{4\pi\hat{\rho}}{k^2\hat{\varphi}}\right)}_{\varepsilon(\vec{k},\omega)} \hat{\varphi} = 0$$

$\varepsilon(\vec{k},\omega) \Rightarrow \varepsilon = 0$ when looking for the normal modes of the system

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NOW:

$$k^2 \varphi_{k\omega} = \underbrace{4\pi q n_{k\omega}} + \overbrace{4\pi q \delta n_{k\omega}}^{\text{discrete particle source response}}$$

usual linear response of plasma to $k^2\hat{\varphi}$

$\delta n_{k\omega}$ is an added background source in which each particle acts as a source

$$\Rightarrow k^2 \underbrace{\varepsilon_{k\omega}} \varphi_{k\omega} = +4\pi e \delta n_{k\omega}$$

dielectric plasma response to $\delta n_{k\omega}$ perturbation ($4\pi q n_{k\omega}$ included within)

RHS no longer zero! $\Rightarrow$ In the case of thermal fluctuations, as in this derivation, we are looking at the response of the system to the discrete particles that are streaming through it. The consequence is that there is an effective driver and $\varepsilon_{k\omega}$ determines the response to this driver. The driver has particular values of ω, k which don't necessarily satisfy $\varepsilon = 0$ when it is present.

↓ write $n_{k\omega}$ as a #-density $\int d\vec{v}$ ↓

$$k^2 \varepsilon_{k\omega} \varphi_{k\omega} = +4\pi e \int d\vec{v}\, \delta \underbrace{\mathcal{F}_{k\omega}(\vec{v})}_{n_{k\omega}\text{'s response to } \varphi_{k\omega}} \qquad \text{eq. } ①$$

where

$$\mathcal{F}(\vec{x},\vec{v},t) = \sum_i \delta(\vec{x}-\vec{x}_i)\delta(\vec{v}-\vec{v}_i)$$

location & velocity of each "source"

** It is crucial to represent particles as discrete in $\mathcal{F}(\vec{x},\vec{v},t)$ to get thermal fluctuations

Aside: Properties of $\mathcal{F}(\vec{x},\vec{v},t)$

$$\iint d\vec{x}\, d\vec{v}\, \mathcal{F}(\vec{x},\vec{v},t) = \iint d\vec{x}\, d\vec{v} \left[\sum_i \delta(\vec{x}-\vec{x}_i)\delta(\vec{v}-\vec{v}_i)\right] = N$$

the total number of particles in the plasma

→

KLE

$\langle \mathcal{F}(\vec{x},\vec{v},t)\rangle = f(\vec{v})$

the standard distribution function

such that

$\langle \mathcal{F}(\vec{x},\vec{v},t)\,\mathcal{F}(\vec{x}',\vec{v}',t')\rangle = \underbrace{f(\vec{v})f(\vec{v}')}_{\text{joint probability in the case of no influence between f's}}$

$+\sum_i \delta(\vec{x}-\vec{x}_i(t))\delta(\vec{x}'-\vec{x}_i(t'))\delta(\vec{v}-\vec{v}_i)\delta(\vec{v}'-\vec{v}_i)$

the <u>2-Particle Distribution Function</u> to describe the influence the particle distributions have on each other (ignoring this yields our standard [collisionless] Vlasov eq. — the "1-particle correlation function")

*This cross-correlation term is what must be included to describe that all particles are weakly correlated.

→ Fourier Transform $\mathcal{F}(\vec{x},\vec{v},t)$ into $\vec{k}$-space

$$\mathcal{F}_k(\vec{v},t) = \sum_i \frac{1}{L^3} e^{-i\vec{k}\cdot\vec{x}_i}\,\delta(\vec{v}-\vec{v}_i)$$

$\vec{x}_i = \vec{x}_{i0} + \vec{v}_i t$

→ Fourier Transform $\mathcal{F}_k(\vec{v},t)$ into ω-space

T here is time, not temperature

$$\mathcal{F}_{k\omega}(\vec{v}) = \sum_i \int_{-T/2}^{T/2} \frac{dt}{T}\frac{1}{L^3} e^{-i\vec{k}\cdot\vec{x}_{i0}} e^{i\omega t} e^{-i\vec{k}\cdot\vec{v}_i t}\,\delta(\vec{v}-\vec{v}_i)$$

$\bar{\omega}_i = \omega - \vec{k}\cdot\vec{v}_i$

$$= \sum_i \int_{-T/2}^{T/2} \frac{dt}{T}\frac{1}{L^3} e^{-i\vec{k}\cdot\vec{x}_{i0}}\, e^{i\bar{\omega}_i t}\,\delta(\vec{v}-\vec{v}_i)$$

$$= \frac{1}{L^3}\sum_i e^{-i\vec{k}\cdot\vec{x}_{i0}}\delta(\vec{v}-\vec{v}_i)\underbrace{\int_{-T/2}^{T/2}\frac{dt}{T}e^{i\bar{\omega}_i t}}$$

$$= (2/2)\,\tfrac{1}{i\bar{\omega}_i t}\left(e^{i\bar{\omega}_i T/2} - e^{-i\bar{\omega}_i T/2}\right)$$

$$= \frac{\sin(\bar{\omega}_i T/2)}{(\bar{\omega}_i T/2)}$$

$$\mathcal{F}_{kw}(\vec{v}) = \frac{1}{L^3}\sum_i e^{-i\vec{k}\cdot\vec{x}_{io}}\,\delta(\vec{v}-\vec{v}_i)\,\frac{\sin(\bar{\omega}_i T/2)}{(\bar{\omega}_i T/2)}$$

↓ Plugging back into eq. ① ↓

$$k^2 \varepsilon_{kw} \varphi_{kw} = 4\pi e \int d\vec{v}\, \delta\left[\frac{1}{L^3}\sum_i e^{-i\vec{k}\cdot\vec{x}_{io}}\,\delta(\vec{v}-\vec{v}_i)\frac{\sin(\bar{\omega}_i T/2)}{(\bar{\omega}_i T/2)}\right]$$

Need to address both + & − ω:

Taking ε_{kw} & ε_{k-w} to other side

$$k^2 \varphi_{k-w} k^2 \varphi_{kw} = k^4 |\varphi_{kw}|^2$$

$\varphi_{k-w}\varphi_{kw} = |\varphi_{kw}|^2$ from reality conditions

$$\begin{cases} \varphi_{-k-w} = \varphi^*_{kw} \\ \varepsilon_{-k-w} = \varepsilon^*_{kw} \end{cases}$$

$$= \frac{1}{L^6} 16\pi^2 e^2 \left\langle \sum_{i,j} \int d\vec{v}\, e^{-i\vec{k}\cdot\vec{x}_{io}} e^{i\vec{k}\cdot\vec{x}_{jo}} \delta(\vec{v}-\vec{v}_i)\delta(\vec{v}-\vec{v}_j) \cdot \frac{1}{|\varepsilon_{kw}|^2}\frac{\sin(\bar{\omega}_i T/2)}{(\bar{\omega}_i T/2)}\frac{\sin(\bar{\omega}_j T/2)}{(\bar{\omega}_j T/2)} \right\rangle \qquad \text{eq. ②}$$

where

$$\langle\ \rangle = \int \frac{d\vec{x}_{io}}{L^3}\int \frac{d\vec{v}_i f(\vec{v}_i)}{n}\int \frac{d\vec{x}_{jo}}{L^3}\int \frac{d\vec{v}_j f(\vec{v}_j)}{n}\cdots$$

$\vec{x},\vec{v}$ space average

· Inspect $\sum_{ij} e^{-i\vec{k}\cdot\vec{x}_{io}} e^{i\vec{k}\cdot\vec{x}_{jo}}$

⇒ $\sum_{ij}$ must = $\underbrace{\sum_i}$ since otherwise $\langle e^{i\vec{k}\cdot\vec{x}_{io}}\rangle = 0$

- $\sum_i \to N$, total # of particles in the whole system
- $\int \frac{1}{L^3} d\vec{x}_{io} \to 1$, conservation of n
- no average over j

then...

$$\text{eq. ②} = \underbrace{\frac{N}{L^6 n}}\int d\vec{v}\, f(\vec{v})\frac{1}{|\varepsilon_{kw}|^2}\underbrace{\frac{\sin^2(\bar{\omega}T/2)}{(\bar{\omega}^2T^2/4)}}$$

$$= \frac{N}{L^6 (N/V)} = 1/L^3$$

$$= \delta(\bar{\omega})\underbrace{\int d\bar{\omega}\frac{\sin^2(\bar{\omega}T/2)}{(\bar{\omega}^2T^2/4)}}_{2\pi/T} = \frac{2\pi}{T}\delta(\bar{\omega})$$

↓ Plugging this all back in ↓

recall: $\bar{\omega} = \omega - kv$

$$k^4 |\varphi_{k\omega}|^2 = \frac{2\pi}{L^3 T} \int d\vec{v} f(\vec{v}) \frac{\delta(\omega - \vec{k}\cdot\vec{v})}{|\varepsilon_{k\omega}|^2} 16\pi^2 e^2$$

$= k^2 (k^2 |\varphi_{k\omega}|^2) = k^2 |E_k|^2$

then

$$\left\langle \frac{|E_{k\omega}|^2}{8\pi} \right\rangle = \frac{k^4 |\varphi_{k\omega}|^2}{8\pi k^2}$$

$$= \frac{2\pi}{L^3 T} \int d\vec{v} f(\vec{v}) \frac{\delta(\omega - \vec{k}\cdot\vec{v})}{|\varepsilon_{k\omega}|^2} \frac{2\pi e^2}{k^2}$$

*from Lecture #8

$$\varepsilon = 1 + \sum_{e,i} \frac{4\pi q^2}{k^2 m} \int d\vec{v} \frac{\vec{k}\cdot \partial f/\partial \vec{v}}{\omega - \vec{k}\cdot\vec{v}} \Rightarrow \varepsilon_{k\omega} = 1 + \frac{4\pi e^2}{k^2 m} \int d\vec{v} \frac{1}{\omega - \vec{k}\cdot\vec{v}} \vec{k}\cdot\frac{\partial f}{\partial \vec{v}}$$

f = Maxwellian

$$\frac{\partial f}{\partial \vec{v}} = \underbrace{\frac{-2\vec{v}}{v_t^2}}_{v_t^2 = 2T_e/m} f = \frac{-\cancel{2} m\vec{v}}{\cancel{2} T_e} f = \frac{-m\vec{v}}{T_e} f$$

$v_t^2 = 2T_e/m$ (electron temp., not time)

$$\varepsilon = 1 + \frac{4\pi e^2}{k^2 \cancel{m}} \left(\frac{-\cancel{m}}{T_e} \right) \int d\vec{v} \frac{\vec{k}\cdot\vec{v} f}{\omega - \vec{k}\cdot\vec{v}}$$

$$= 1 - \left(\frac{4\pi e^2}{k^2 T_e} \frac{n_0}{n_0} \right) \int d\vec{v} \frac{\vec{k}\cdot\vec{v} f}{\omega - \vec{k}\cdot\vec{v}}$$

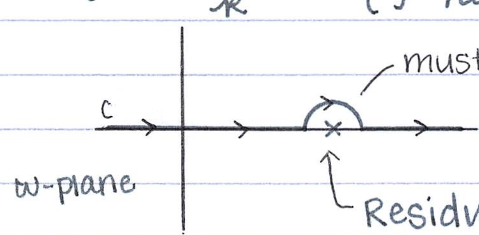
insert

$k_{De}^2 = 4\pi n_0 e^2 / T_e$

$$= 1 - \frac{k_D^2}{k^2} \int \frac{d\vec{v} f}{n_0} \frac{\vec{k}\cdot\vec{v}}{\omega - \vec{k}\cdot\vec{v}}$$

↓ Inspect imaginary component to resolve residue ↓

$$\mathrm{Im}\{\varepsilon\} = \frac{-k_{De}^2}{k^2} \mathrm{Im}\left\{ \int \frac{d\vec{v} f}{n_0} \frac{\omega}{\omega - \vec{k}\cdot\vec{v}} \right\}$$



must close contour from above

Residue = $-i\pi$ (CW integration)

$$\mathrm{Im}\{\varepsilon\} = \frac{+k_{De}^2}{k^2}\,\mathrm{Im}\left\{\int \frac{d\vec{v} f}{n_0}(+i\pi)\,\delta(\omega - \vec{k}\cdot\vec{v})\right\}$$

$$= \frac{k_D^2}{k^2}\pi\omega \int \frac{d\vec{v} f}{n_0}\,\delta(\omega - \vec{k}\cdot\vec{v})$$

$$\hookrightarrow \int d\vec{v} f\,\delta(\omega - \vec{k}\cdot\vec{v}) = \mathrm{Im}\{\varepsilon\}\frac{k^2 n_0}{k_{De}^2 \pi\omega}$$

↓ Plugging back in for $\langle |E|^2/8\pi \rangle$ ↓

$$\left\langle \frac{|E_{k\omega}|^2}{8\pi} \right\rangle = \frac{2\pi}{L^3 T}\int d\vec{v} f(\vec{v})\frac{\delta(\omega - \vec{k}\cdot\vec{v})}{|\varepsilon_{k\omega}|^2}\frac{2\pi e^2}{k^2}$$

$$= \frac{2\pi}{L^3 T}\,\mathrm{Im}\{\varepsilon\}\frac{\cancel{k^2} n_0}{\underbrace{k_D^2}_{=4\pi n_0 e^2/T_e}\pi\omega}\frac{1}{|\varepsilon_{k\omega}|^2}\frac{2\pi e^2}{\cancel{k^2}}$$

$$= \frac{2\pi}{L^3 T}\frac{\mathrm{Im}\{\varepsilon\}}{|\varepsilon_{k\omega}|^2}\frac{\cancel{n_0}}{\pi/\omega}\frac{\cancel{2\pi}\cancel{e^2}T_e}{{}^2\cancel{4\pi}\cancel{n_0}\cancel{e^2}}$$

$$= -\mathrm{Im}\left\{\frac{1}{\varepsilon_{k\omega}}\frac{1}{\omega}\right\}$$

Thermal Fluctuation Levels:

$$\Rightarrow L^3\left\langle \frac{|E_{k\omega}|^2}{8\pi} \right\rangle = -\frac{2\pi}{T}\frac{1}{\pi}\,\mathrm{Im}\left\{\frac{1}{\varepsilon_{k\omega}}\frac{1}{\omega}\right\}\frac{T_e}{2} \qquad \text{eq. (3)}$$

T_e ← temp; T ← time

** Note that from this we see that even in thermal equilibrium, there are electric field fluctuations in any plasma

Besides discreteness in k, we also have discreteness in ω!

$d\omega = \frac{2\pi}{T}$, T = time

would disappear if integrated over ω

↳ ω is strictly real here (it's a parameter), so we write the imaginary component from ε

$$\mathrm{Im}\{\varepsilon\} = \frac{k_{De}^2}{k^2}\pi\omega \int d\vec{v}\frac{f}{n_0}\underbrace{\delta(\omega - \vec{k}\cdot\vec{v})}_{\text{resonance condition}}$$

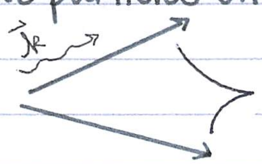
Two parts give you ε_I:

- Growth rate
- Resonance (intrinsic imaginary part)

$$\int d\vec{v} \underbrace{\frac{1}{(\vec{v}-\omega/k)}}_{\text{denominator gives resonance}} \partial f/\partial \vec{v}$$

$\Rightarrow$ source of fluctuations is a resonance!

We have particles streaming around (electrons & ions)



streaming such that $\vec{k}\cdot\vec{v}$ matches ω yields resonance with the wave (resonating e^-)

Want to integrate eq. ③ over ω at a given $\vec{k}$-value since $\vec{k}$ defines a degree of freedom:

ω is discrete $\rightarrow$ must start with a summation!

$$\sum_\omega L^3 \left\langle \frac{|E_{k\omega}|^2}{8\pi} \right\rangle = -\underbrace{\sum_\omega \frac{2\pi}{T}}\frac{1}{\pi}\,\mathrm{Im}\left\{\frac{1}{\varepsilon_{k\omega}}\frac{1}{\omega}\right\}\frac{T_e}{2}$$

$$\sum_\omega \frac{2\pi}{T} = \int d\omega$$

discrete $\rightarrow$ continuous

$$= -\int \frac{d\omega}{\pi}\frac{T_e}{2}\,\mathrm{Im}\left\{\frac{1}{\varepsilon_{k\omega}}\right\}\frac{1}{\omega} = -\frac{1}{\pi}\frac{T_e}{2}\int\frac{d\omega}{\omega}\,\mathrm{Im}\left\{\frac{1}{\varepsilon_{k\omega}}\right\}$$

$$\hookrightarrow L^3\left\langle \sum_\omega \frac{|E_{k\omega}|^2}{8\pi}\right\rangle = \frac{-1}{\pi}\frac{T_e}{2}\int\frac{d\omega}{\omega}\,\mathrm{Im}\left\{\frac{1}{\varepsilon_{k\omega}}\right\}$$

$\hookrightarrow$ Is there a singularity @ $\omega = 0$ for $1/\omega\,\mathrm{Im}\{1/\varepsilon_{k\omega}\}$?

Need to ask: Is there an imaginary part of ε as $\omega \to 0$?

$$\lim_{\omega\to 0} \varepsilon_{k\omega} = 1 + \frac{k_D^2}{k^2}$$

slow process = no waves (all you get is Debye shielding

$\Rightarrow$ There is no imaginary component in Debye shielding, so $\mathrm{Im}\{1/\varepsilon\} = 0$ as $\omega \to 0$

$\hookrightarrow$ No singularity here!

This allows us to pull Im{} out of our integral

$$L^3 \left\langle \sum_\omega \frac{|E_{k\omega}|^2}{8\pi} \right\rangle = \frac{-1}{\pi} \frac{T_e}{2} \, \mathrm{Im}\left\{ \underbrace{P}_{\text{principal value integral}} \int \frac{d\omega}{\omega} \frac{1}{\varepsilon_{k\omega}} \right\}$$

where...

The principal value integral may be broken down as

$$P\int d\omega = \int_{\delta}^{\infty} d\omega\,(\;) + \int_{-\infty}^{\delta} d\omega\,(\;)$$

δ skips $\omega = 0$ value

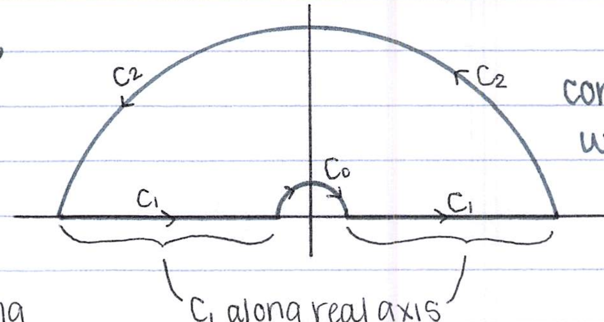
Consider:

$$\oint_C \frac{d\omega}{\omega} \underbrace{\left(\frac{1}{\varepsilon} - 1\right)}_{} = 0$$ — no singularity in contour

At high frequencies, $\varepsilon \to 1$, so this cancels perfectly @ $\omega \to \infty$

$\varepsilon = 1 - \omega_{pe}^2/\omega^2$ for really large ω, the plasma does not even have time to react to the wave, so $\varepsilon \to 1$

With a contour defined as



complex ω-plane

*C_2 corresponds to large ω; we want this to cancel out, so we added the -1 term in our $\oint_C$

- keep in mind: There is no singularity within our above-defined contour. A singularity in the UHP would mean a growing wave, which cannot occur in thermodynamic equilibrium for lack of free energy $\Rightarrow$ All poles of ε must be in the LHP (all waves damped)

$$\oint_C = \underbrace{\int_{C_2} \frac{d\omega}{\omega}(\;)}_{\text{this goes to 0}} + \int_{C_0} \frac{d\omega}{\omega}(\;) + \underbrace{\int_{C_1} \frac{d\omega}{\omega}(\;)}_{\text{this is what we want}}$$

$$= \int_{C_0} \frac{d\omega}{\omega}\left(\frac{1}{\varepsilon} - 1\right) + \int_{C_1} \frac{d\omega}{\omega}\left(\frac{1}{\varepsilon} - 1\right) = 0$$

Principal value integral

$$\underbrace{\int_{C_1} \frac{d\omega}{\omega}\left(\frac{1}{\varepsilon}-1\right)}_{\text{this is the integral w/o waves}} = -\underbrace{\int_{C_0} \frac{d\omega}{\omega}\left(\frac{1}{\varepsilon}-1\right)}_{\text{Residue} = -i\pi \text{ (CW integration)}}$$

$$= i\pi\left[\underbrace{\frac{1}{(1+k_D^2/k^2)}}_{\varepsilon|_{\omega=0}} - 1\right]$$

$$= i\pi\left[\left(\frac{1}{1+k_D^2/k^2}\right) - \left(\frac{1+k_D^2/k^2}{1+k_D^2/k^2}\right)\right] = i\pi\left(\frac{-k_D^2/k^2}{1+k_D^2/k^2}\right)\left(\frac{k^2/k_D^2}{k^2/k_D^2}\right) \text{ insert}$$

$$= -i\pi \frac{1}{1+k^2/k_D^2}$$

then...

$$L^3\left\langle \sum_\omega \frac{|E_{k\omega}|^2}{8\pi}\right\rangle = -\frac{1}{\pi}\frac{T_e}{2}\underbrace{\operatorname{Im}\left\{P\int \frac{d\omega}{\omega}\left(\frac{1}{\varepsilon_{k\omega}}-1\right)\right\}}$$

$$= \operatorname{Im}\left\{P\int \frac{d\omega}{\omega}\left(\frac{1}{\varepsilon}-1\right)\right\} + \cancelto{0}{\operatorname{Im}\left\{P\int \frac{d\omega}{\omega}\right\}}$$

our $\int_{C_0}$ result

$$= \frac{-1}{\pi}\frac{T_e}{2}\operatorname{Im}\left\{\frac{-i\pi}{1+k^2/k_D^2}\right\}$$

$$\Rightarrow L^3\left\langle \sum_\omega \frac{|E_{k\omega}|^2}{8\pi}\right\rangle = \frac{T_e}{2}\frac{1}{1+k^2/k_D^2}$$ → breaks down at short wavelengths; "catastrophic failure"

* Each degree of freedom of k has $T_e/2$ energy ($\frac{1}{2}kT$ for each d.o.f. for each value of k)

Sum over $\vec{k}$ → Isotropic result

$$\left\langle \sum_\omega \sum_{k} \frac{|E_{k\omega}|^2}{8\pi}\right\rangle = \frac{1}{(2\pi)^3}\frac{T_e}{2}\sum_k (2\pi)^3 \frac{1}{L^3}\frac{1}{1+k^2/k_D^2}$$

insert; $\sum_k \left(\frac{2\pi}{L}\right)^3 = \int d\vec{k}$

summing over $\vec{k}$ gives the total wave energy in the fluctuations

$$\left\langle \sum_{\omega}\sum_{k} \frac{|E_{k\omega}|^2}{8\pi} \right\rangle = \frac{T_e}{2(2\pi)^3} \underbrace{\int d\vec{k}\, \frac{1}{1+k^2/k_D^2}}_{\int_0^\infty \frac{4\pi k^2 d\vec{k}}{1+k^2/k_D^2}}$$

Behavior:
- $\rightarrow 0$ for large $\vec{k}$-values
- Divergence in the integral for small $\vec{k}$-values
  - └ something must keep this from having infinite energy, but what?

Wave-Particle Interactions {G&R Ch. 25}

This issue is directly linked to resonances!

* Plasma particles can exchange energy with waves even in the absence of classical dissipative processes through resonance interactions with waves *

- For a collisionless plasma with no classical dissipation, can you have irreversible interaction between particles & waves?
  - ↳ Yes! Resonant interactions between particles and waves yields, e.g., irreversible energy exchange
    - * Resonance is the only way to get irreversible processes within a plasma (note that this issue is only well-defined in the absence of collisions)
- How do we describe these resonant interactions?

i.e., How do we describe the exchange of energy beyond just our previous linearity approximation?

⇒ Particles moving close to the phase speed of the wave don't see an oscillatory field — see a quasi-DC field — and therefore can give or take energy from the wave (resonant)

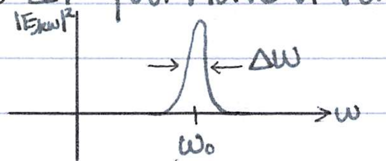
⇒ Particles moving with very different velocities from v_p see an oscillatory field and there is typically no irreversible energy exchange (non-resonant)

KLE

Resonant interactions take two distinct forms depending on the nature of the wave spectrum:

(both give irreversible energy exchange)

① If you have a very narrow wave spectrum



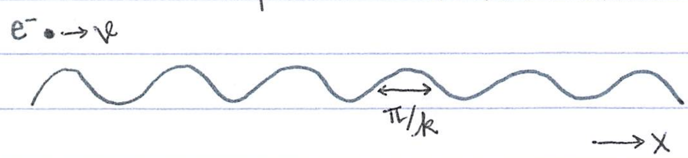
very narrow peak @ ω_0

If $\Delta\omega = 0$, particle would see the wave oscillating forever

This very narrow spectrum has nearly constant frequency, so the particle sees it for a long time → not just a pulse like what would be seen for a broad wave spectrum

⇒ This large timeframe for interaction allows for <u>particle trapping</u> by the wave

- Consider a particle [electron] moving in the frame of the wave:
  - the wave potential is static/stationary in the wave frame (no time-dependence in this frame)



$\varphi(x) = \varphi_0 \cos(kx)$

In this frame, the energy of the particle is conserved

$\frac{1}{2}mv^2 - e\varphi = W$ = particle energy = constant

$\frac{1}{2}mv^2 = W + e\varphi$

$= \underbrace{W + e\varphi_0 \cos(kx)}$

⇒ for $W < e\varphi_0$ particles are trapped since $v^2 \to 0$

(particle MUST be reflected because we cannot have $v^2 < 0$)

How do we describe this trapping?

↳ Expand potential about $x = 0$ $\left[\cos(ax) \simeq 1 - \frac{(ax)^2}{2!} + \frac{(ax)^4}{4!} - \ldots\right]$

keep 1st O only

- near $x=0$

$W = \frac{1}{2}mv^2 - e\varphi_0\left(1 - \frac{1}{2}k^2x^2\right)$

multiply on both sides

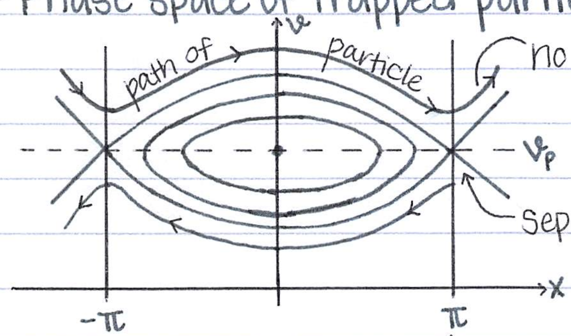
$$\left(\frac{2}{m}\right)(W+e\varphi_0) = \frac{1}{2}m\left(v^2 + \frac{e\varphi_0 k^2}{m}x^2\right)\left(\frac{2}{m}\right)$$

$$\underbrace{\frac{2}{m}(W+e\varphi_0)}_{\text{If this is identically}=0,} = \underbrace{v^2 + \overbrace{\frac{k^2}{m}}^{\text{constant}} e\varphi_0 x^2}_{\text{Yields an ellipse in } x\text{-}v \text{ space!}}$$

If this is identically $= 0$,

that corresponds to $(x=0,\ v=v_p) \rightarrow$ ellipse central point

$\Rightarrow$ Phase space of trapped particle motion



• Characteristic time associated with this:

i.e. the bounce time of deeply trapped particles

$$v_0^2 = v^2 + \frac{k^2}{m}e\varphi_0 x^2$$

$v_0^2 = \frac{2}{m}(W+e\varphi_0)$, a constant

$\rightarrow$ taking the time derivative to find the equation of motion...

$$0 = \cancel{2v}\underbrace{\dot{v}}_{\ddot{x}} + \frac{k^2}{m}e\varphi_0 \cancel{2}x\cancel{\dot{x}}$$

($\dot{x} = v$)

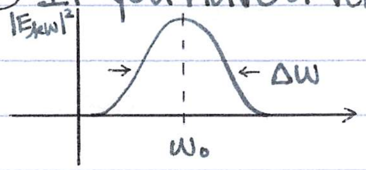
$$\Rightarrow \ddot{x} + \underbrace{\frac{k^2}{m}e\varphi_0}_{\omega_b^2 \text{ — the bounce frequency}} x = 0 \text{ , a harmonic oscillator!}$$

$$\omega_b = \sqrt{\frac{k^2 e\varphi_0}{m}}$$

* Note: the bounce timescale cannot be described by linear theory

→ From this we can see that $\omega_b \gg \Delta\omega$ is required for particle trapping

② If you have a very broad wave spectrum



Here, $\Delta\omega \gg \omega_b$ → the particle sees many waves and sees coherent acceleration only for a short time

- Correlation time
  - time in which the particle will see this wave

$$\tau_c = \frac{2\pi}{\Delta\omega}$$ — $\Delta\omega$ = spectral width

⇒ If $\omega_b \tau_c \ll 1$, there will be no particle trapping (no time for a "bounce"). Instead we get <u>random interactions</u> of particles with waves — a series of small "kicks" as the waves come in and leave.

How do we describe this series of kicks?

↳ The energy exchange of the particles with waves can be treated using a statistical approach

*often the nonlinearities of the system can then be considered weak and one can expand the equations of motion in powers of wave amplitude

⇒ Quasilinear Theory!

We cannot describe resonant interaction ① with linear theory, but resonant interaction ② may be estimated as such

- when there is particle trapping, the statistical approach does not work; motion is then fully nonlinear