Lecture 12 - Wave Energy & Momentum 10/05/17

Continuing from last time...

Dielectric Tensor

$$\underbrace{\left[\left(\frac{\vec{k}\vec{k}}{k^2} - \overline{\overline{I}}\right) + \overline{\overline{\varepsilon}}\right]}\cdot\hat{E} = 0$$

$$\Rightarrow \overline{\overline{G}} = \frac{c^2}{\omega^2}\left(\vec{k}\vec{k} - \overline{\overline{I}}k^2\right) + \overline{\overline{\varepsilon}}$$

$$\overline{\overline{G}}\cdot\hat{E} = 0$$

where

$$\overline{\overline{\varepsilon}} = \varepsilon_{\parallel}\frac{\vec{k}\vec{k}}{k^2} + \varepsilon_{\perp}\left(\overline{\overline{I}} - \frac{\vec{k}\vec{k}}{k^2}\right)$$

$$\varepsilon_{\perp} = 1 + \frac{\omega_{pe}^2}{\omega^2}\zeta_e Z\underbrace{(\zeta_e)}$$

$$\varepsilon_{\parallel} = 1 - \frac{k_{De}^2}{2k^2}Z'\left(\frac{\omega}{kv_{te}}\right) \qquad \zeta_e = \frac{\omega}{kv_{te}}$$

such that

det$|\overline{\overline{G}}| = 0$ gives the General Dispersion Relation

*describes both electrostatic & electromagnetic waves

Wave Energy & Momentum

From Maxwell's Equations we know that

$$\frac{\partial}{\partial t}U + \nabla\cdot\vec{S} = \underbrace{-\vec{E}\cdot\vec{J}}_{\text{work done by the field on the particles}}$$

↳ <0 : it's a loss term

where

$$U = \frac{E^2 + B^2}{8\pi}, \text{ energy}$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}, \text{ the Poynting Vector}$$

↳ gives the magnitude & direction of transport of energy through a surface

KLE

We also have a momentum equation:

$$\frac{d}{dt}\left(\vec{p}_{particles} + \vec{p}_{field}\right) = \nabla \cdot \overline{\overline{T}}$$

$\vec{p}_{field} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \vec{S}/c^2$

$\vec{p}_{particles} = \sum_j m_j n_j \vec{v}_j$

$\overline{\overline{T}}$: Maxwell stress tensor — a rank 2 tensor used in classical electromagnetism to represent the interaction between electromagnetic forces and mechanical momentum

→ We want to use these equations to evaluate energy and momentum as we previously did for electrostatic waves

Energy Density of a General Wave

ASK: How much work do I do to create a wave of some amplitude?

i.e., how much work do I put in in building up a wave?

→ Introduce some test charge and test current

$\rho_t, \vec{J}_t$

If we insert $\rho_t, \vec{J}_t$ into Poisson's & Maxwell's equations and build them up, we can find out how much energy was put into the wave.

- Poisson

$$\nabla\cdot\vec{E} = 4\pi\rho$$
$$= 4\pi\rho_{plasma} + 4\pi\rho_t$$

- Maxwell

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$= \frac{4\pi}{c}\vec{J}_{plasma} + \frac{4\pi}{c}\vec{J}_t + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

We can relate the test elements with the charge continuity eqn.

$\nabla\cdot\vec{D} = \rho_{free} \rightarrow \nabla\cdot\frac{\partial}{\partial t}\vec{D} = \frac{\partial}{\partial t}\rho_{free} = -\nabla\cdot\vec{J}_{free}$ (from PHYS606)

$\frac{\partial}{\partial t}\rho_t + \nabla\cdot\vec{J}_t = 0$, Continuity of test charge

↓ Fourier transform ↓

$-i\omega_k\rho_{kt} + i\vec{k}\cdot\vec{J}_{kt} = 0$

where our "reality conditions" from Lec. #10 must still be met

$$\omega_k = \omega_R + i\gamma_k$$

γ_k — Landau damping arising from resonant particles

$$\omega_{-k} = -\omega_k^*$$
$$= \underbrace{-\omega_R}_{\text{wholly real}} + i\gamma_k$$

Rate at which energy enters the wave:

$$\dot{W} = -\underbrace{\langle \vec{J}_t \cdot \vec{E} \rangle}_{\text{space-averaged}}$$

$$= -(\text{work done by the field on } \vec{J}_t)$$

$$= -\int \frac{d\vec{x}}{L^3} \vec{J}_t \cdot \vec{E} \qquad \vec{E} = -\nabla\varphi, \quad \varphi = \text{Re}\{\varphi_k e^{i\vec{k}\cdot\vec{x} - i\omega t}\}$$

$$= +\int \frac{d\vec{x}}{L^3} \vec{J}_t \cdot \left[+ \sum_k \nabla \text{Re}\left\{ \varphi_k e^{i\vec{k}\cdot\vec{x} - i\omega_k t} \right\} \right]$$

$$\vec{J} = \int d^3v\, q\vec{v}\hat{f}, \quad \hat{f} = \frac{-q}{m}\varphi_{-k} \frac{-\vec{k}\cdot \overbrace{\partial/\partial\vec{v} f_0}^{\frac{\partial}{\partial \vec{v}} f_0 = \frac{-2\vec{v}}{v_{th}^2} f_0}}{(-\vec{k}\cdot\vec{v} - \omega_{-k})}$$

$$= +\int \frac{d\vec{x}}{L^3} \int d^3v \frac{q^2\vec{v}}{m} \sum_{k,k'} \varphi_{-k} \frac{+\vec{k}\cdot}{(-\vec{k}\cdot\vec{v} - \omega_{-k})} \left(\frac{-2\vec{v}}{v_{th}^2} \right) f_0$$

$$e^{i\vec{k}\cdot\vec{x} - i\omega_{-k}t} \cdot \nabla \varphi_{-k} e^{i\vec{k}\cdot\vec{x} - i\omega_k t}$$

↓ Skipping a ton of steps shown in Lec. #10 ↓

$$= -i\frac{q^2}{m} \sum_{-k} \int d\vec{v} \frac{\omega_{-k}}{\omega_{-k} + (-\vec{k})\cdot\vec{v}} (-\vec{k}) \cdot \frac{-2\vec{v}}{v_{th}^2} f_0 e^{2\gamma_k t} |\varphi_k|^2$$

$$= -\sum_k \vec{J}_{tk} \cdot \underbrace{\vec{E}_{-k}}_{} e^{2\gamma_k t}$$

"reality conditions", Lec #10

$$\vec{E}_k = -i\vec{k}\varphi_k \longrightarrow \vec{E}_{-k} = i\vec{k}\varphi_{-k} = i\vec{k}\varphi_k^*$$
$$= \vec{E}_k^*$$

$$\dot{W} = -\sum_k \underbrace{\vec{J}_{tk}}_{\text{need to evaluate this}} \cdot \vec{E}_k^* e^{2\gamma_k t} \qquad \text{eq. } ①$$

⇒ Solve our Maxwell's eqn. for J_t and plug in

KLE

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}_p + \frac{4\pi}{c}\vec{J}_t + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}$$

$\nabla \to i\vec{k}$ $\quad\quad$ $\frac{\partial}{\partial t} \to -i\omega$

$$i\vec{k}\times\hat{\vec{B}} = \frac{4\pi}{c}\hat{\vec{J}}_p + \frac{4\pi}{c}\hat{\vec{J}}_t - \frac{1}{c}i\omega\hat{\vec{E}}$$

$\hookrightarrow \hat{\vec{B}} = \frac{c}{\omega}\vec{k}\times\hat{\vec{E}}$ as found in Lec. #11

$$i\vec{k}\times\left(\frac{c}{\omega}\vec{k}\times\hat{\vec{E}}\right) = \frac{4\pi}{c}\hat{\vec{J}}_p + \frac{4\pi}{c}\hat{\vec{J}}_t - \frac{1}{c}i\omega\hat{\vec{E}}$$

Substitute the Dielectric tensor

$\frac{4\pi}{c}\hat{\vec{J}} - \frac{i\omega}{c}\hat{\vec{E}} = \frac{-i\omega}{c}\overline{\overline{\varepsilon}}\cdot\hat{\vec{E}}$ (Lec. #11)

$$\frac{4\pi}{c}\hat{\vec{J}}_t = i\vec{k}\times\left(\frac{c}{\omega_k}\vec{k}\times\vec{E}_k\right) + i\frac{\omega_k}{c}\overline{\overline{\varepsilon}}\cdot\vec{E}_k$$

$$= \frac{i\omega_k}{c}\overline{\overline{G}}\cdot\vec{E}_k$$

↓ plug back into eq. ① ↓

$$\Rightarrow \dot{W} = -\sum_k \frac{i\omega_k}{4\pi}\vec{E}_k^*\cdot\overline{\overline{G}}\cdot\vec{E}_k e^{2\gamma_k t} \quad\quad \text{eq. ②}$$

Assume small damping (maximal build-up)

$\hookrightarrow$ expand $\overline{\overline{G}}$

$\omega_k = \omega_R + i\gamma_k$

$$\omega_k\overline{\overline{G}} \sim \omega_R\overline{\overline{G}}(\omega_R) + i\gamma_k\left(\frac{\partial}{\partial\omega_R}\omega_R\overline{\overline{G}}(\omega_R)\right)$$

no significant contribution to $\overline{\overline{G}}(\omega_k)$ from γ_k term as we've neglected dissipation from Landau resonance

SO: $\mathrm{Im}\{\overline{\overline{G}}(\omega_R)\} = 0$

$\Rightarrow \overline{\overline{G}}_{-k} = \overline{\overline{G}}_k$

↓ plug expansion into eq. ② ↓

goes to 0 when summed over $\pm k$ since $\overline{\overline{G}}_{-k} = \overline{\overline{G}}_k$

$$\dot{W} = -\sum_k \frac{i}{4\pi}\vec{E}_k^*\cdot\left[\cancel{\omega_R\overline{\overline{G}}(\omega_R)} + i\gamma_k\frac{\partial}{\partial\omega_R}\left(\omega_R\overline{\overline{G}}(\omega_R)\right)\right]\cdot\vec{E}_k e^{2\gamma_k t}$$

$$= \sum_k (2)\gamma_k\frac{1}{(2)4\pi}\vec{E}_k^*\cdot\frac{\partial}{\partial\omega_R}\left[\omega_R\overline{\overline{G}}(\omega_R)\right]\cdot\vec{E}_k \underbrace{e^{2\gamma_k t}}$$

insert

only dependence on t

$$\frac{d}{dt}e^{2\gamma_k t} = 2\gamma_k e^{2\gamma_k t}$$

KLE

$$\dot{W} = \sum_k \frac{\vec{E}_k^*}{8\pi}\frac{\partial}{\partial\omega_R}\left[\omega_R \overline{\overline{G}}(\omega_R)\right]\cdot \underbrace{\frac{\partial}{\partial t}\vec{E}_k}$$

where we've pulled $e^{2\gamma_k t}$ into our definition for $\vec{E}_k$ — see Lecture #10 for full derivation

$$\Rightarrow W = \sum_k \frac{\vec{E}_k^*}{8\pi}\frac{\partial}{\partial\omega_R}\left[\omega_R \overline{\overline{G}}(\omega_R)\right]\cdot \vec{E}_k$$

Plug back in for $\overline{\overline{G}}$

$$\overline{\overline{G}}\cdot\vec{E}_k = \vec{k}\times(\vec{k}\times\vec{E}_k)\,c^2/\omega_k^2 + \overline{\overline{\varepsilon}}\cdot\vec{E}_k$$

$$W = \sum_k \frac{\vec{E}_k^*}{8\pi}\frac{\partial}{\partial\omega_R}\left[\omega_R\left(\underbrace{\vec{k}\times(\vec{k}\times\vec{E}_k)\frac{c^2}{\omega_k^2}}_{\text{Magnetic field energy, } |B_k|^2\ (\hat{B}=\frac{c}{\omega}\vec{k}\times\hat{E})} + \overline{\overline{\varepsilon}}\cdot\vec{E}_k\right)\right]$$

↓ multiply the $\vec{E}_k^*$ term through ↓

$$= \sum_k \frac{1}{8\pi}\left[|B_k|^2 + |E_k|^2 + \underbrace{\vec{E}_k^*\frac{\partial}{\partial\omega_R}\left(\omega_R(\overline{\overline{\varepsilon}} - \overline{\overline{I}})\cdot\vec{E}_k\right)}_{\text{Plasma Sloshing Energy (the plasma energy)}}\right]$$

Momentum Density

We want to calculate the rate of momentum transfer from the test current to the wave.

Force density of an electromagnetic field:

$$(f) = \rho\vec{E} + \underbrace{\frac{1}{c}\vec{J}}\times\vec{B}$$

assuming the test current density has a direction parallel to $\vec{v}_t$ and a magnitude $\rho_t|\vec{v}_t|$...

$$\vec{J}_t = \rho_t\vec{v}_t$$

$$(f) = \rho_t\vec{E} + \frac{1}{c}\rho_t\vec{v}_t\times\vec{B}$$

$\rho_t = q_t n_t = e_t n_t$

$$= e n_t\vec{E} + \frac{1}{c}e n_t\vec{v}_t\times\vec{B}$$

*Net force on a differential volume element dV of the fluid is

$$d\vec{F} = d(d\vec{p}/dt) = (f)\,dV$$

$\Rightarrow$ Space average yields $\vec{F} = d\vec{p}/dt$

KLE

$$\vec{F}_t = \frac{d}{dt}\vec{p}_t = \langle (f) \rangle$$

$$= \langle \underbrace{e n_t}_{\rho_t} \vec{E} + \frac{1}{c} \overbrace{e n_t \vec{v}_t}^{\vec{J}_t} \times \vec{B} \rangle$$

$$\dot{\vec{p}}_t = \langle \rho_t \vec{E} + \frac{1}{c} \vec{J}_t \times \vec{B} \rangle$$

each is an infinite sum in k-space

ex: $\langle \rho_t \vec{E} \rangle$

$$= \langle \sum_k \rho_{tk} e^{i\vec{k}\cdot\vec{x} - i\omega_k t} \sum_{k'} E_{k'} e^{i\vec{k}'\cdot\vec{x} - i\omega_{k'} t} \rangle$$

↓ express average as integral over $\vec{x}$ ↓

$$\int \frac{d^3x}{L^3} \underbrace{e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{\delta\text{-function}} = \underbrace{\frac{(2\pi)^3}{L^3}}_{\times \sum_{k'} \to \int dk'} \delta(\vec{k}+\vec{k}')$$

1; ω's cancel for $k' = -k$

$$\langle \rho_t \vec{E} \rangle = \sum_k \int dk' \underbrace{\delta(\vec{k}+\vec{k}')}_{\text{This Dirac-}\delta\text{ requires that } k' = -k} \rho_{tk} E_{k'} \, \cancel{e^{-i(\omega_k + \omega_{k'})t}}$$

$$= \sum_k \rho_{tk} \vec{E}_{-k}$$

$\vec{E}_{-k} = \vec{E}^*_k$ from our reality conditions

→ Break down $\dot{\vec{p}}_t$ into a sum over all values of k
(addresses all wave components)

$$\dot{\vec{p}}_t = \sum_k \left[E^*_k \rho_{tk} + \frac{1}{c} \vec{J}_{tk} \times \vec{B}^*_k \right]$$

(from Faraday's law: $\vec{B}_k = \frac{c}{\omega_k}(\vec{k} \times \vec{E}_k)$)

$$= \sum_k \left[\vec{E}^*_k \rho_{tk} + \underbrace{\frac{1}{\cancel{c}} \vec{J}_{tk} \times \frac{1}{\cancel{c}} \left(\frac{\cancel{c}}{\omega_k^*} (\vec{k} \times \vec{E}^*_k) \right)}_{} \right]$$

$$\vec{J}_{tk} \times (\vec{k} \times \vec{E}^*_k) \frac{1}{\omega^*_k} = \frac{\vec{k}}{\omega^*_k} \vec{J}_{tk} \cdot \vec{E}^*_k - \frac{\vec{k} \cdot \vec{J}_{kt}}{\omega^*_k} \vec{E}^*_k$$

$$= \sum_k \left[\vec{E}^*_k \rho_{tk} + \frac{\vec{k}}{\omega^*_k} \vec{J}_{tk} \cdot \vec{E}^*_k - \underbrace{\frac{\vec{k} \cdot \vec{J}_{tk}}{\omega^*_k}}_{} \vec{E}^*_k \right]$$

from charge continuity equation:

$$-\omega_k \rho_{tk} + \vec{k} \cdot \vec{J}_{tk} = 0$$

$$\hookrightarrow \vec{J}_{tk} = \omega_k / \vec{k} \, \rho_{tk}$$

$$= \sum_{\vec{k}} \left[\vec{E}^*_k \rho_{tk} + \frac{\vec{k}}{\omega^*_k} \vec{J}_{tk} \cdot \vec{E}^*_k - \frac{\vec{k}}{\omega^*_k} \frac{\omega_k}{\vec{k}} \rho_{tk} \vec{E}^*_k \right]$$

$(\omega_R + i\gamma_k)/(\omega_R - i\gamma_k) \simeq 1$; γ_k small

$$= \sum_{\vec{k}} \left[\cancel{\vec{E}^*_k \rho_{tk}} + \frac{\vec{k}}{\omega^*_k} \vec{J}_{tk} \cdot \vec{E}^*_k - \cancel{\rho_{tk} \vec{E}^*_k} \right]$$

$$\Rightarrow \dot{\vec{p}}_t \simeq \sum_{\vec{k}} \frac{\vec{k}}{\omega^*_k} \underbrace{\vec{J}_{tk} \cdot \vec{E}^*_k e^{2\gamma_k t}}$$ from reality conditions on ω_k, as before

recognize form of eq. ①

$$\dot{W} = -\sum_{\vec{k}} \underbrace{\vec{J}_{tk} \cdot \vec{E}^*_k e^{2\gamma_k t}}_{=\dot{W}_k}$$

$$= -\sum_{\vec{k}} \frac{\vec{k}}{\omega^*_k} \dot{W}_k$$

Then, since $\dot{\vec{p}}_t = -\dot{\vec{p}}_{wave}$

$\Rightarrow \vec{p}_{wave} = \sum_{\vec{k}} \frac{\vec{k}}{\omega_k} W_k \longrightarrow \vec{p}_{plasma} = \frac{\vec{k}}{\omega_k} W_{plasma}$

The momentum of the wave is directly related to the energy content of the wave

But how are power & information propagated through waves?

$\hookrightarrow$ must introduce group velocity

Group Velocity & Power Flux

We start with our general dispersion relation

$\det|\overleftrightarrow{G}| = 0$

$\hookrightarrow$ this yields ω_k

We look to prove from this that:

- Group Velocity

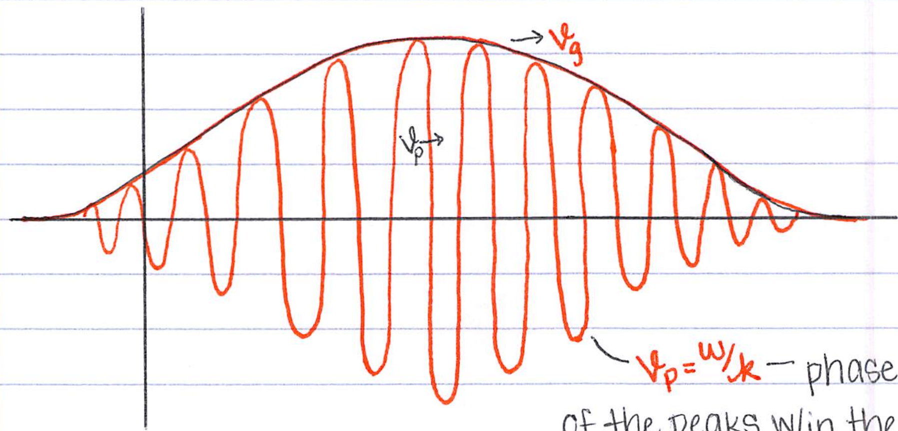
  $\vec{v}_g = \frac{d\omega_k}{d\vec{k}}$ — velocity of a wave packet — rate of energy transport

and

- Energy Flux [Power Flux]

  $\vec{\Gamma} = \vec{v}_g W \sim \text{ergs}/\text{s}\cdot\text{cm}^2$

KLE



$v_p = \omega/k$ — phase velocity is the velocity of the peaks w/in the pulse (envelope)

To show all this, we need to address the behavior of a wave packet.

Consider

$\overline{\overline{G}} \cdot \vec{E} = 0$ [for all k]

expand out in Taylor series, taking δ's to be small

Let

$\omega = \omega_0 + \delta\omega$

$k = k_0 + \delta k$

$\vec{E} = \vec{E}_0 + \delta\vec{E}$

where $\delta\omega, \delta k$ correspond to slow time and space variation

Expand in $\delta\omega, \delta\vec{k}$; with $\overline{\overline{G}}_0 = \overline{\overline{G}}(\omega_0, \vec{k}_0)$

$$\underbrace{\cancel{\overline{\overline{G}}_0 \cdot \hat{E}_0}}_{0,\ \overline{\overline{G}} \cdot \vec{E} = 0} + \overline{\overline{G}}_0 \cdot \delta\hat{E} + \delta\omega \frac{\partial}{\partial\omega_0}\underbrace{(\overline{\overline{G}}_0 \cdot \hat{E}_0)} + \delta\vec{k} \cdot \frac{\partial}{\partial k_0}\underbrace{(\overline{\overline{G}}_0 \cdot \hat{E}_0)} = 0$$

leave these

Take the dot-product from the left with $\hat{E}_0^*$

↳ assume $\overline{\overline{G}}$ is Hermitian (self-adjoint)

$$\underbrace{\cancel{\hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \delta\hat{E}}}_{0,\text{ since } \overline{\overline{G}} \text{ is Hermitian}} + \delta\omega \frac{\partial}{\partial\omega_0}(\hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \hat{E}_0) + \delta\vec{k} \frac{\partial}{\partial k_0}(\hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \hat{E}_0)$$

$\frac{\partial}{\partial\omega_0}, \frac{\partial}{\partial k_0}$ only act on $\overline{\overline{G}}_0$, so we can bring this inside

$$\Rightarrow \delta\omega \frac{\partial}{\partial\omega_0}(\hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \hat{E}_0) = -\delta\vec{k} \frac{\partial}{\partial k_0}(\hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \hat{E}_0)$$

Multiply everything by ω_0

└ bring it inside derivatives since $\overline{\overline{G}}_0 \cdot \hat{E}_0 = 0$

KLE

$$\underbrace{(-i)\,\delta\omega \frac{\partial}{\partial \omega_0}\left(\frac{\omega_0}{(8\pi)}\hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \hat{E}_0\right)}_{\text{insert}}^{W,\text{ energy}} + \underbrace{(+i)\,\delta\vec{k}\frac{\partial}{\partial \vec{k}_0}\left(\frac{-\omega_0}{(8\pi)}\hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \hat{E}_0\right)}_{\text{insert}}^{\vec{\Gamma},\text{ energy flux}} = 0$$

Then $\Rightarrow -i\delta\omega W + i\delta\vec{k}\cdot\vec{\Gamma} = 0$

$$\hookrightarrow \vec{\Gamma} = \frac{-\omega_0}{8\pi}\frac{\partial}{\partial \vec{k}_0}\left(\hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \hat{E}_0\right)$$

Let

$$\Sigma \equiv \omega_0 \hat{E}_0^* \cdot \overline{\overline{G}}_0 \cdot \hat{E}_0$$

such that

$$\frac{d}{d\vec{k}}\Sigma\bigg|_{k_0,\omega_0} = \frac{\partial}{\partial \vec{k}}\Sigma\bigg|_{k_0} + \frac{\partial}{\partial \omega_0}\Sigma \overbrace{\frac{d\omega_0}{d\vec{k}}\bigg|_{k_0}}^{\text{Group velocity}}$$

$= 0$ must be zero since Σ is zero for all k-values, i.e., constant

$$\rightarrow \frac{d\omega_k}{d\vec{k}} = \vec{v}_g = \frac{-\partial\Sigma/\partial\vec{k}_0}{\partial\Sigma/\partial\omega_0} = \text{the velocity @ which the wave envelope is moving}$$

$$\frac{\partial \Sigma}{\partial \vec{k}_0} = -\vec{v}_g \frac{\partial \Sigma}{\partial \omega_0}$$

Then, plugging in...

$$\vec{\Gamma} = \frac{+\omega_0}{8\pi}\left[\frac{+\partial\Sigma}{\partial\omega_0}\vec{v}_g\right] = \vec{v}_g \underbrace{\frac{\omega_0}{8\pi}\frac{\partial\Sigma}{\partial\omega_0}}_{W}$$

$$\vec{\Gamma} = \vec{v}_g W$$

$\Rightarrow$ Wave energy propagates at the group velocity

$$V_g = \frac{1}{k}\sqrt{k^2c^2 + \omega_p^2} = \frac{1}{k}\omega(k)$$

Energy Flux

$$\vec{\Gamma} = \frac{-\omega_0}{8\pi}\frac{\partial}{\partial \vec{k}_0}\vec{E}_0^* \cdot \overline{\overline{G}} \cdot \vec{E}_0 \qquad \text{eq. } 3$$

$$\hookrightarrow \overline{\overline{G}} = \frac{c^2}{\omega^2}\vec{k}\times\underbrace{(\vec{k}\times\ )}_{\text{operates on what } \overline{\overline{G}} \text{ is dotted with}} + \overline{\overline{\varepsilon}} \overbrace{- \overline{\overline{I}} + \overline{\overline{I}}}^{\text{insert}}$$

$\longrightarrow$

KLE

$$\frac{\partial}{\partial \vec{k}_0}\vec{E}_0^*\cdot\overline{\overline{G}}\cdot\vec{E}_0=\frac{\partial}{\partial \vec{k}_0}\vec{E}_0^*\cdot\left[\frac{c^2}{\omega^2}\vec{k}\times(\vec{k}\times\ )+\overline{\overline{\varepsilon}}-\overline{\overline{I}}+\overline{\overline{I}}\right]\cdot\vec{E}_0$$

$$=\frac{\partial}{\partial \vec{k}_0}\left[-\underbrace{(\vec{k}\times\vec{E}_0)\cdot(\vec{k}\times\vec{E}_0^*)}\right]$$

two infinite wave vectors – assign separate notation

$$=-2\,\partial/\partial\vec{k}'\,(\vec{k}'\times\vec{E}_0)\cdot(\vec{k}\cdot\vec{E}_0^*)=-2\vec{E}_0\times(\vec{k}\times\vec{E}_0^*)\cdot\underbrace{\partial/\partial\vec{k}'\,\vec{k}'}_{\overline{\overline{I}}}$$

$$=-2\vec{E}_0\times(\vec{k}\times\vec{E}_0^*)$$

↓plugging into eq. ③↓

$$\vec{\Gamma}=\frac{-\omega_0}{8\pi}\frac{\partial}{\partial \vec{k}_0}\vec{E}_0^*\cdot\overline{\overline{G}}\cdot\vec{E}_0=\frac{-\omega_0}{8\pi}\frac{\partial}{\partial \vec{k}_0}\vec{E}_0^*\cdot\left[\frac{c^2}{\omega^2}\vec{k}\times(\vec{k}\times\ )+\overline{\overline{\varepsilon}}-\overline{\overline{I}}+\overline{\overline{I}}\right]\cdot\vec{E}_0$$

$$=\frac{+\cancel{\omega_0}}{4\cancel{8}\pi}\left[\cancel{+2}\,\underbrace{\vec{E}_0\times(\vec{k}\times\vec{E}_0^*)\Big]\frac{c^2}{\omega_0^{\cancel{2}}}}_{\vec{B}_0=c/\omega_0\,\vec{k}\times\vec{E}_0}-\frac{\omega_0}{8\pi}\frac{\partial}{\partial \vec{k}_0}\vec{E}_0^*\cdot(\overline{\overline{\varepsilon}}-\overline{\overline{I}})\cdot\vec{E}_0$$

$$=\underbrace{c\frac{\vec{E}_0\times\vec{B}_0^*}{4\pi}}_{\vec{S}}-\underbrace{\frac{\omega_0}{8\pi}\frac{\partial}{\partial \vec{k}_0}\vec{E}_0^*\cdot(\overline{\overline{\varepsilon}}-\overline{\overline{I}})\cdot\vec{E}_0}_{\text{particles}}$$

Poynting Flux

particles — At any given location there is energy passing from the EM field to the particles, but on average (over space) both EM energy (Poynting Flux) and particle energy propagate together as a unit.

In the cold plasma limit:

$$\overline{\overline{\varepsilon}}=\overline{\overline{I}}\left(1-\frac{\omega_{pe}^2}{\omega_0^2}\right)$$

↓plug into $\vec{\Gamma}$↓

$$\frac{\partial}{\partial \vec{k}_0}(\overline{\overline{\varepsilon}}-\overline{\overline{I}})=0\Rightarrow\vec{\Gamma}=c\frac{\vec{E}_0\times\vec{B}_0^*}{4\pi}-0$$

Only Poynting Flux in the cold Plasma limit