Lecture 11 - Adding in Electromagnetic Waves 10/03/17

from last time...

Wave Energy: Electrostatic Waves

Energy content of a wave

$$W_{wave} = \frac{|E_k|^2}{8\pi} \frac{\partial}{\partial \omega_k}\left[\omega_k \varepsilon(k, \omega_k)\right] \rightarrow$$ particle sloshing energy plus electric field energy

*does NOT include energy of the wave interacting with resonant particles

In getting to the above, we used:

- Reality Conditions -

where $\varphi(\vec{x},t) = \varphi_k e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + \varphi_{-k} e^{i\vec{k}\cdot\vec{x} - i\omega_{-k} t}$ MUST = wholly real

- $\varphi_{-k} = \varphi_k^*$
- $\omega_{-k} = -\omega_k^*$ $\rightarrow$ The sign of k does not affect damping or growth; it's the slope $\partial f/\partial \vec{v}|_{v_\varphi}$ that determines that

& from linear theory

$\hookrightarrow \chi_{-k} = \chi_k^*$

$$\dot{W}_p = -i \underbrace{\sum_k}_{\text{this is a sum over } + \& - k} \omega_k \chi_k |E_k|^2$$

$$= \omega_k \chi_k + \omega_{-k}\chi_{-k}$$

$$= \omega_k \chi_k - \omega_k^* \chi_k^*$$ $\rightarrow$ real component is zero, must extract imaginary component

$Im\{\omega_k \chi_k\}$ has two contributions

- imaginary part of χ
- $\omega = \omega_R + i\gamma_k$

$$= Im\left\{\omega_R i \chi_I + \underbrace{\frac{\partial}{\partial \omega_k}(\omega_k \chi)\Big|_{\omega_k}}_{=\chi_R} i\gamma_k\right\}$$

then

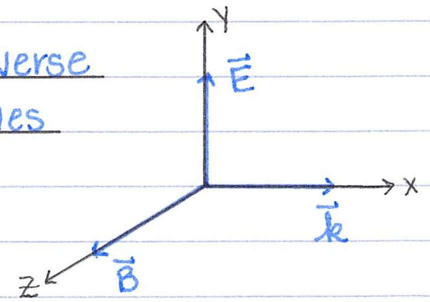
$$\dot{W}_p = \sum_k \left[\frac{\partial}{\partial \omega_k}(\omega_k \chi)\Big|_{\omega_k} 2\gamma_k + \underbrace{2\omega_R \chi_I}_{\text{resonant stuff, which we dropped}}\right]\frac{|E_k|^2}{2}$$

KLE

Electromagnetic Waves

*No ambient magnetic field, but the wave itself produces a field

Transverse Waves



Maxwell's Equations

- $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}$
- $\frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \nabla\times\vec{E} = 0$

We need the current, $\vec{J}$

→ start with the linearized Vlasov eqn.

$$-i\omega\hat{f} + i\vec{k}\cdot\vec{v}\hat{f} + \frac{q}{m}\Big(\hat{\vec{E}} + \underbrace{\frac{1}{c}\vec{v}\times\hat{\vec{B}}}_{\text{new contribution from generated } \vec{B}\text{-field}}\Big)\cdot\frac{\partial}{\partial\vec{v}}f_0 = 0$$

BUT! This goes away since $\partial/\partial\vec{v}\, f_0 \sim \vec{v}$, f_0 Maxwellian

$$(\vec{v}\times\vec{B})\cdot\frac{\partial}{\partial\vec{v}} = \frac{\partial}{\partial\vec{v}}\cdot(\vec{v}\times\vec{B})$$
$$= \vec{B}\cdot\underbrace{\left(\frac{\partial}{\partial\vec{v}}\times\vec{v}\right)}_{0} = 0$$

This leaves us with the distribution function

$$\hat{f} = \frac{-q}{m}\frac{\hat{\vec{E}}\cdot\partial/\partial\vec{v}\, f_0}{i(\vec{k}\cdot\vec{v}-\omega)}$$

previously, we've been writing this w.r.t. $\hat{\varphi} = i\hat{E}/k$

Use this to find $\vec{J}$

$$\hat{\vec{J}} = \underbrace{\int d^3v\, q\vec{v}\hat{f}}_{=qn\vec{u}}$$

$$= \frac{-q^2}{m}\int d^3v\,\vec{v}\,\frac{\hat{\vec{E}}\cdot\partial/\partial\vec{v}\, f_0}{i(\vec{k}\cdot\vec{v}-\omega)}$$

but f_0 is Maxwellian

$$f_0 = \frac{n_0}{\pi^{3/2}v_t^3}e^{-v^2/v_t^2} \longrightarrow \frac{\partial}{\partial\vec{v}}f_0 = \frac{-2\vec{v}}{v_t^2}f_0$$

$$= \frac{+q^2}{m}\int d^3v\,\vec{v}\,\frac{(+2/v_t^2)\hat{\vec{E}}\cdot\vec{v}f_0}{i(\vec{k}\cdot\vec{v}-\omega)}$$

$\hat{\vec{E}} \sim \hat{y}$

$\vec{k} \sim \hat{x}$

$$\hat{\vec{J}} = \frac{2q^2}{m v_t^2}\int d^3v\, \frac{\vec{v}\,\hat{E}_y v_y f_0}{i(k_x v_x - \omega)}$$

→ only term that survives is in the y-direction ⤵

$$\hat{J}_y = \frac{2q^2}{m v_t^2}\int d^3v\, \frac{v_y^2 f_0 \hat{E}_y}{i(k_x v_x - \omega)}$$

$$= \frac{2q^2}{m v_t^2}\int dv_x \int dv_z \underbrace{\int dv_y \frac{v_y^2 \hat{E}_y}{i(k_x v_x - \omega)} \frac{n_0}{\pi^{3/2} v_t^3} e^{-\vec{v}^2/v_t^2}}_{=\frac{v_t^2}{2}\frac{f_0 \hat{E}_y}{i(k_x v_x - \omega)}}$$

$$= \frac{\cancel{2}q^2}{m \cancel{v_t^2}}\int dv_x \underbrace{\int dv_z}_{\text{gives } (\sqrt{\pi}\, v_{th})^{-1}} \frac{\cancel{v_t^2}}{\cancel{2}} \frac{\hat{E}_y}{i(k_x v_x - \omega)} \frac{n_0}{(\pi^{3/2} v_t^3)} e^{-\vec{v}^2/v_t^2}$$

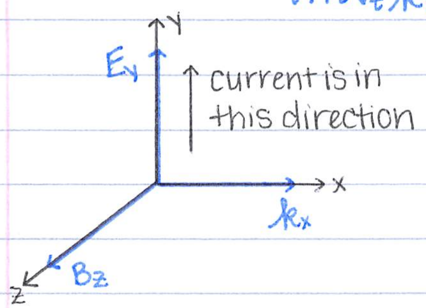
$$= \frac{n_0 q^2}{k_x m i}\hat{E}_y \underbrace{\int \frac{dv_x}{\sqrt{\pi}\, v_t} \frac{1}{v_x - \omega/k_x} e^{-v_x^2/v_t^2}}$$

recognize this as a Z-function (ref: Lec. #9)

$$= Z\left(\frac{\omega}{k_x v_t}\right) \to \zeta = \frac{\omega}{k_x v_t}$$

defined for $\mathrm{Im}\{\zeta\} > 0$

$$\Rightarrow \hat{\vec{J}} = \hat{J}_y \hat{y} = \hat{y}\, \frac{q^2 \hat{E}_y n_0}{i m v_t k_x} Z\left(\frac{\omega}{k_x v_t}\right)$$



Plug back into Maxwell's Equations
(assuming wave-form)

$$\underbrace{\nabla\times}_{\nabla \to i\vec{k}} \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\underbrace{\frac{\partial}{\partial t}}_{\partial/\partial t \to i\omega}\vec{E}$$

$$\begin{cases} i\vec{k}\times\hat{\vec{B}} = \dfrac{4\pi}{c}\dfrac{q^2}{m}\dfrac{n_0}{i k_x v_t} Z\left(\dfrac{\omega}{k_x v_t}\right)\hat{\vec{E}} - \dfrac{i\omega}{c}\hat{\vec{E}} \\ -\dfrac{i\omega}{c}\hat{\vec{B}} + i\vec{k}\times\hat{\vec{E}} = 0 \end{cases}$$

$\hookrightarrow \hat{\vec{B}} = \frac{c}{\omega}\vec{k}\times\hat{\vec{E}}$

→

Substituting for $\hat{\vec{B}}$ in the first equation...

$$\cancel{i}\vec{k}\times\left(\underbrace{\frac{c}{\omega}}_{\text{divide through}}\vec{k}\times\hat{\vec{E}}\right)=\frac{4\pi}{c}\frac{q^2}{m}\frac{-n_0}{\cancel{i}k_x v_t}Z(\zeta)\hat{\vec{E}}-\frac{\cancel{i}\omega}{c}\hat{\vec{E}}$$

$\omega_{pe}^2 = 4\pi n_0 q^2/m$

$$\underbrace{\vec{k}\times(\vec{k}\times\hat{\vec{E}})}_{-k^2\hat{\vec{E}}}=\left[\frac{-4\pi}{c^2}\frac{q^2}{m}\frac{n_0}{(k_x v_t)}\omega\, Z(\zeta)-\frac{\omega^2}{c^2}\right]\hat{\vec{E}}$$

$\zeta = \omega/k_x v_t$

$$+k^2\hat{\vec{E}}=\left[\frac{+\omega_{pe}^2}{c^2}\zeta Z(\zeta)+\underbrace{\frac{\omega^2}{c^2}}_{\text{factor out } \omega^2/c^2}\right]\hat{\vec{E}}$$

$$\Rightarrow k^2\hat{\vec{E}}=\frac{\omega^2}{c^2}\varepsilon_\perp\hat{\vec{E}}$$

$$\varepsilon_\perp = 1+\frac{\omega_{pe}^2}{\omega^2}\zeta_e Z(\zeta_e)$$

For $\hat{\vec{E}}\neq 0$ we require:

$$k^2\cancel{\hat{\vec{E}}}=\omega^2/c^2\,\varepsilon_\perp\cancel{\hat{\vec{E}}}$$

$$\frac{k^2c^2}{\omega^2}=\varepsilon_\perp=1+\frac{\omega_{pe}^2}{\omega^2}\zeta_e Z(\zeta_e)$$

$$\omega^2=k^2c^2-\omega_{pe}^2\zeta_e Z(\zeta_e)$$

{ Fun fact: Phase speed of these waves exceeds the speed of light }

↳ This is okay! v_g, the group velocity, the velocity at which the wave envelope is moving, must be < c since it carries both information and energy. v_p, the phase velocity, is the velocity of the peaks within the wave pulse (envelope).

- energy associated with the field is propagating at the pulse velocity

$$v_g=\sqrt{k^2c^2+\omega_p^2}/k=\omega(k)/k$$

Look @ the large argument limit : $\zeta = \omega / k v_t \gg 1$

$$Z(\zeta) = \int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}} \frac{e^{-s^2}}{s - \zeta}$$

$$s \equiv \frac{v_x^2}{v_t^2}, \quad \zeta \equiv \frac{\omega}{k v_t}$$

$$\lim_{\zeta \gg 1} Z(\zeta_e) \simeq \frac{-1}{\zeta_e}$$

then...

$\varepsilon_{\parallel}$ in the cold plasma limit

$$\varepsilon_{\perp} = 1 + \frac{\omega_{pe}^2}{\omega^2} \zeta_e Z(\zeta_e) \longrightarrow 1 + \frac{\omega_{pe}^2}{\omega^2} \cancel{\zeta_e} \left(\frac{-1}{\cancel{\zeta_e}} \right) = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

$$\omega^2 = k^2 c^2 - \omega_{pe}^2 \zeta_e Z(\zeta_e) \longrightarrow \underbrace{k^2 c^2 + \omega_{pe}^2}$$

$\Rightarrow \omega^2 = k^2 c^2 + \omega_{pe}^2$ Dispersion relation for Electromagnetic waves in a plasma

What does this mean?

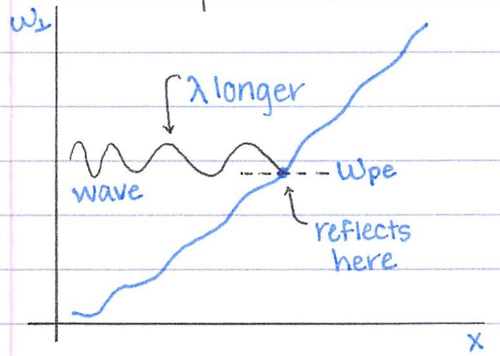
- Light waves propagate anywhere $k^2 c^2 > 0$
  - ↳ that is, $\omega^2 - \omega_{pe}^2 > 0$ or $\omega > \omega_{pe}$ required for propagation
- Cut off for $\omega < \omega_{pe}$; $k^2 c^2 = \omega^2 - \omega_{pe}^2 < 0$

  *note that this means that ham radios can communicate world wide as you can bounce signals off the ionosphere (known as "skip")

If negative ($k^2 c^2 < 0$) or $\omega < \omega_{pe}$ → EM waves are reflected from regions in which $\omega < \omega_{pe}$

• These are evanescent waves, which are oscillating electric and/or magnetic fields that decay in space but not in time

└ fields penetrate the medium but power does not



⇒ You can bounce waves off the ionosphere for low enough frequency waves

[Better @ night — plasma boundary sharper]

KLE

→ As $k \to 0$, phase speed → ∞ (but that's okay — group velocity is still < c and no information is propagating)

→ No Landau damping for these waves

NOW: Put Electrostatic & Electromagnetic together
(still no ambient $\vec{B}$-field)

General Dielectric Function

Begin by rewriting Maxwell's equations:

$$i\vec{k}\times\hat{\vec{B}} = \frac{4\pi}{c}\hat{\vec{J}} - \frac{i\omega}{c}\hat{\vec{E}} \equiv \frac{-i\omega}{c}\overline{\overline{\varepsilon}}\cdot\hat{\vec{E}}$$

Dielectric tensor

$$\vec{k}\times\hat{\vec{B}} = \frac{-\omega}{c}\overline{\overline{\varepsilon}}\cdot\hat{\vec{E}}$$

and from before — $\hat{\vec{B}} = \frac{c}{\omega}\vec{k}\times\hat{\vec{E}}$

Put these together

$$\vec{k}\times\left(\frac{c}{\omega}\vec{k}\times\hat{\vec{E}}\right) = \frac{-\omega}{c}\overline{\overline{\varepsilon}}\cdot\hat{\vec{E}}$$

$$c^2/\omega^2\,\underbrace{\vec{k}\times(\vec{k}\times\hat{\vec{E}})}_{=(\vec{k}\vec{k}-\overline{\overline{I}}k^2)\cdot\hat{\vec{E}}} + \overline{\overline{\varepsilon}}\cdot\hat{\vec{E}} = 0 \qquad \text{eq. (1)}$$

↓ dot equation w/ $\vec{k}$ ↓

$$c^2/\omega^2\,\vec{k}\cdot(\vec{k}\vec{k}-\overline{\overline{I}}k^2)\cdot\hat{\vec{E}} + \vec{k}\cdot\overline{\overline{\varepsilon}}\cdot\hat{\vec{E}} = 0$$

$$\Rightarrow \vec{k}\cdot\overline{\overline{\varepsilon}}\cdot\hat{\vec{E}} = 0$$

where

$$\varepsilon_{\parallel} = 1 - \frac{k_{De}^2}{2k^2}Z'\left(\frac{\omega}{kv_{te}}\right), \text{ given}$$

* derived in Lecture #10

** Cold plasma limit

$$\varepsilon_{\parallel} = \varepsilon_{\perp} = 1 - \omega_{pe}^2/\omega^2$$

Returning to eq. (1)

$$c^2/\omega^2\,\vec{k}\times(\vec{k}\times\hat{\vec{E}}) + \overline{\overline{\varepsilon}}\cdot\hat{\vec{E}} = 0$$

$$\underbrace{c^2/\omega^2}_{1/k^2}(\vec{k}\vec{k}-\overline{\overline{I}}k^2)\cdot\hat{\vec{E}} + \overline{\overline{\varepsilon}}\cdot\hat{\vec{E}} = 0$$

$$\left(\frac{\vec{k}\vec{k}}{k^2} - \overline{\overline{I}}\right)\cdot\hat{E} = -\overline{\overline{\varepsilon}}\cdot\hat{E}$$

From this we can write

given above

$$\overline{\overline{\varepsilon}} = \varepsilon_{\parallel}\frac{\vec{k}\vec{k}}{k^2} + \varepsilon_{\perp}\left(\overline{\overline{I}} - \frac{\vec{k}\vec{k}}{k^2}\right) \Rightarrow \text{satisfies } \vec{k}\cdot\overline{\overline{\varepsilon}}\cdot\hat{E} = 0$$

$$\varepsilon_{\perp} = 1 + \underbrace{\frac{\omega_{pe}^2}{\omega^2}\zeta_e Z(\zeta_e)}_{}$$

what we just found taking the large ζ_e limit

Combining these:

$$\underbrace{\left[\left(\frac{\vec{k}\vec{k}}{k^2} - \overline{\overline{I}}\right) + \overline{\overline{\varepsilon}}\right]}_{}\cdot\hat{E} = 0$$

$$\Rightarrow \overline{\overline{G}} = \frac{c^2}{\omega^2}\left(\vec{k}\vec{k} - \overline{\overline{I}}k^2\right) + \overline{\overline{\varepsilon}}, \text{ where } \overline{\overline{G}}\cdot\hat{E} = 0$$

General Dispersion Relation: $\det|\overline{\overline{G}}| = 0$

↳ Both ES, EM waves described here