Lecture 10 – Wave Energy

09/28/17

from last time...

Damping of Plasma Waves

$$\gamma = \frac{\delta\omega}{i} = \frac{\pi}{2}\omega_{pe}\frac{k}{|k|}\frac{v_p^2}{n_0}\underbrace{\frac{\partial f_0}{\partial v_x}\bigg|_{v_p=\omega_{pe}/k_x}}_{\text{slope (negative)}}$$

* This could also represent growth if the distribution function has the opposite sign

Plasma Dispersion Function (for electrostatic plasma waves)

A standard function representing the kinetic plasma dispersion relation for a Maxwellian distribution

↳ a.k.a. the Z-function

$$Z(\zeta) = \int_{-\infty}^{\infty}\frac{ds}{\sqrt{\pi}}\frac{e^{-s^2}}{s-\zeta}$$

· defined as written for $\mathrm{Im}\{\zeta\}>0$ and is analytically continued for $\mathrm{Im}\{\zeta\}\leq 0$

for $\vec{k} = k_x\hat{i}$

$$\left\{ s \equiv \frac{v_x}{v_t}, \quad \zeta \equiv \frac{\omega}{k_x v_t} \right.$$

a tabulated function in the complex plane

→ The dielectric function may be written with respect to Z

$$\varepsilon = 1 + \frac{k_D^2}{k^2}\left(1+\zeta Z(\zeta)\right)$$

If we differentiate Z with respect to ζ...

$$Z'(\zeta) = \int_{-\infty}^{\infty}\frac{ds}{\sqrt{\pi}}\frac{e^{-s^2}}{(s-\zeta)^2}$$

↓ integrate by parts ↓

$$= \frac{-1}{\sqrt{\pi}}\int_{-\infty}^{\infty}ds\frac{2s}{(s-\zeta)}e^{-s^2}$$

$$= \underbrace{-2\left[1-\zeta Z(\zeta)\right]}_{\text{need to show this in HW \#4}}$$

→

KLE

KLE

Then, in terms of this differential

$$\varepsilon = 1 - \frac{k_D^2}{2k^2} Z'\Big(\underbrace{\frac{\omega}{kv_t}}_{\text{argument; } \zeta = \omega/kv_t}\Big)$$

- For a large argument, $\zeta \gg 1$

$$\frac{1}{s-\zeta} = \frac{-1}{\zeta - s} = \frac{-1}{\zeta}\,\frac{1}{1-\underbrace{s/\zeta}_{\text{small} \to \text{expand}}}$$

* $\omega/kv_{th} \gg 1$, so we're dealing with a cool plasma, high phase velocity

$$= \frac{-1}{\zeta}\left[1 + \frac{s}{\zeta} + \frac{s^2}{\zeta^2} + \dots\right]$$

Plugging this in...

odd → 0

$$Z(\zeta) = -\int_{-\infty}^{\infty} \frac{ds}{\sqrt{\pi}}\,\frac{e^{-s^2}}{\zeta}\left[1 + \cancel{\frac{s}{\zeta}} + \frac{s^2}{\zeta^2} + \dots\right]$$

$$= \frac{-1}{\zeta}\left[1 + \frac{1}{2\zeta^2} + \frac{3}{4}\frac{1}{\zeta^4} + \dots\right]$$

and

0 — now odd because of 2s term out front

$$Z'(\zeta) = \frac{+1}{\sqrt{\pi}}\int_{-\infty}^{\infty} ds\,\frac{2s}{\zeta}\,e^{-s^2}\left[\cancel{1} + \frac{s}{\zeta} + \cancel{\frac{s^2}{\zeta^2}} + \dots\right]$$

$$= \frac{1}{\zeta^2} + \frac{3}{2}\frac{1}{\zeta^4} + \dots$$

→ How does this expansion affect the dielectric function?

$$\varepsilon = 1 - \frac{k_D^2}{2k^2} Z'(\zeta)$$

$$= 1 - \frac{\overbrace{k_D^2}^{k_D^2 = 4\pi n q^2/T}}{2k^2}\,\underbrace{\frac{1}{\zeta^2}}_{\zeta^2 = \omega^2/k^2 v_t^2}\left[1 + \frac{3}{2}\frac{1}{\zeta^2}\right]$$

*note the wave-like response in this limit

KLE

$$\varepsilon = 1 - \frac{4\pi n_0 e^2}{2k^2 T}\frac{k^2 v_{th}^2}{\omega^2}\left(1+\frac{3}{2}\frac{k^2 v_{th}^2}{\omega^2}\right)$$

$$= 1 - \underbrace{\frac{4\pi n_0 e^2}{\cancel{2T}}\frac{\cancel{2T}}{m_e}}_{=\omega_{pe}^2}\frac{1}{\omega^2}\left(1+\frac{3}{2}\frac{k^2 v_{th}^2}{\omega^2}\right)$$

$$= 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1+\frac{3}{2}\frac{k^2 v_{th}^2}{\omega^2}\right) = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1+\frac{3k^2 T}{m_e \omega^2}\right)$$

$\Rightarrow$ The same result we attained when we accounted for thermal corrections in Lecture #8

- For a small argument, $\zeta \ll 1$

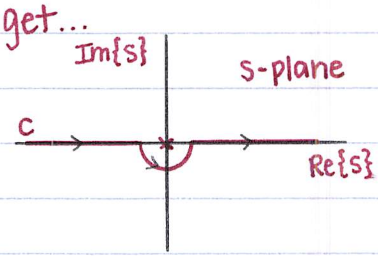
$\rightarrow$ Now the opposite is true; $\omega / k v_{th} \ll 1$, so we're dealing with a warm plasma, small phase velocity

(what you want for e^- in a sound wave)

$$Z(\zeta) \approx Z(0) + (Z'(0))\zeta + \ldots$$

$\hookrightarrow$ if we let ζ tend to zero from the upper half of the complex plane, we get...

$$Z(0) = \int_C \frac{ds}{\sqrt{\pi}}\frac{e^{-s^2}}{s}$$



$$= \underbrace{\frac{i\pi}{\sqrt{\pi}}}_{\text{from semi-circle}} + P\int \frac{ds}{\sqrt{\pi}}\frac{e^{-s^2}}{s}$$

principal value integral

this is ZERO – the integrand is an odd function in s

$$= i\sqrt{\pi}$$

$$Z'(0) = -2\left[1 + 0 \cdot Z(0)\right]$$
$$= -2$$

$\downarrow$ plugging into our expansion for small ζ $\downarrow$

$$Z(\zeta) \approx i\sqrt{\pi} - 2\zeta$$

$\llcorner$ breaking this into components for ζ = real:

$\mathrm{Im}\{Z\} = \sqrt{\pi}\, e^{-\zeta^2}$; from deforming down around singularity

→ Again inspecting the dielectric function

$$\varepsilon = 1 - \frac{k_0^2}{2k^2} Z' = 1 + \frac{k_0^2}{\cancel{2}k^2}(+\cancel{2})$$

$$= 1 + \underbrace{\frac{k_0^2}{k^2}}_{\text{just the Debye shielding response}} \Rightarrow \text{electron Boltzmann response}$$

* this is all in the limit where $\omega / k v_t << 1$

↳ so the distribution will be Maxwellian

↳ no frequency – slow response

Wave Energy – Electrostatic Waves

We want to discuss Landau damping in terms of energy transfer, but first we need to discuss how a wave carries energy in a plasma

Returning to the Vlasov equation

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla + \frac{q}{m} \vec{E} \cdot \frac{\partial}{\partial \vec{v}} \right) f = 0$$

Expand f as a power series in E (as with Krook's model, Lecture #6)

$$f = \overbrace{f_0}^{0^{th} \text{ order in E}} + \underbrace{f_1}_{1^{st} \text{ order in E, etc.}} + f_2 + \ldots$$

To find the change in energy, look @ 2^{nd} order in E

↳ $\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) f_2 + \frac{q}{m} \vec{E} \cdot \frac{\partial}{\partial \vec{v}} f_1 = 0$

- perform a space average $\langle \ \rangle = \int \frac{d^3x}{L^3}$ [∇ term → 0]

$$\underbrace{\frac{\partial}{\partial t} \langle f_2 \rangle + \frac{q}{m} \langle \vec{E} \cdot \frac{\partial}{\partial \vec{v}} f_1 \rangle = 0}_{\text{this will now be independent of space}}$$

→

⇒ **Want to calculate particle energy!**

general form — $\int d\vec{v}\, \frac{1}{2} m v^2 (\ )$

$$\underbrace{E}_{\text{call it } W_p} = \int d^3v \left(\tfrac{1}{2} m v^2\right) \langle f_2 \rangle$$

$$\dot{W}_p + \frac{q}{\cancel{m}} \left\langle \int d^3v \, \frac{1}{2} \cancel{m} v^2 \, \overbrace{\vec{E}}^{\vec{E} = -\nabla\varphi} \cdot \frac{\partial}{\partial \vec{v}} f_1 \right\rangle = 0$$

↓ integrate by parts ↓

$$\dot{W}_p + q \underbrace{\left\langle \int d^3v f_1 \vec{v} \cdot \nabla\varphi \right\rangle}_{\text{convert space average to integral form}} = 0$$

$$\dot{W}_p = -q \int \frac{d\vec{x}}{L^3} \int d\vec{v} f_1 \vec{v} \cdot \nabla\varphi$$

- from Lecture #7 (electrostatic waves in warm plasma; $\varepsilon(\omega_k, k) = 0$)

$f_1 = \mathrm{Re}\{\hat{f} e^{i\vec{k}\cdot\vec{x} - i\omega_k t}\}$

$$\hat{f} = -\frac{q}{m} \underbrace{\hat{\varphi}}_{\hat{\varphi} = \varphi_k} \frac{1}{\omega - \vec{k}\cdot\vec{v}} \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0$$

$$\rightarrow f_1 = -\sum_k e^{i\vec{k}\cdot\vec{x} - i\omega_k t} \frac{q}{m} \varphi_k \frac{1}{\omega_k - \vec{k}\cdot\vec{v}} \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0$$

$$\varphi = \mathrm{Re}\{\underbrace{\hat{\varphi}}_{\hat{\varphi} = \varphi_{k'}} e^{i\vec{k}\cdot\vec{x} - i\omega t}\}$$

$$\rightarrow \vec{v} \cdot \nabla\varphi = \sum_{k'} i\vec{k'} \cdot \vec{v} \varphi_k e^{i\vec{k'}\cdot\vec{x} - i\omega_k t}$$

plugging into $\dot{W}_p$...

$$\dot{W}_p = +q \int \frac{d\vec{x}}{L^3} \int d\vec{v} \left(+\sum_k e^{i\vec{k}\cdot\vec{x} - i\omega_k t} \frac{q}{m} \frac{\varphi_k}{\omega_k - \vec{k}\cdot\vec{v}} \vec{k} \cdot \frac{\partial}{\partial \vec{v}} f_0 \right) \left(\sum_{k'} i\vec{k'} \cdot \vec{v} \varphi_{k'} e^{i\vec{k'}\cdot\vec{x} - i\omega_{k'} t} \right)$$

the 2nd order contributions are a product of f_1 & φ. Each is a sum over all values of k. The product of f_1, φ is therefore a product of two distinct infinite sums, k & k'

⟶

KLE

$$\dot{W}_p = i\frac{q^2}{m}\sum_{\vec{k},\vec{k}'}\int\frac{d\vec{x}}{L^3}\int d\vec{v}\,\underbrace{e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}}\,e^{-i(\omega_k+\omega_{k'})t}\frac{\vec{k}'\cdot\vec{v}}{\omega_k-\vec{k}\cdot\vec{v}}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0\varphi_k\varphi_{k'}$$

$$\int d\vec{x}\exp[i(\vec{k}+\vec{k}')\cdot\vec{x}] = (2\pi)^3\delta(\vec{k}+\vec{k}')$$

$$= i\frac{q^2}{m}\underbrace{\sum_{\vec{k},\vec{k}'}\frac{(2\pi)^3}{L^3}}\int d\vec{v}\,\delta(\vec{k}+\vec{k}')e^{-i(\omega_k+\omega_{k'})t}\frac{\vec{k}'\cdot\vec{v}}{\omega_k-\vec{k}\cdot\vec{v}}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0\varphi_k\varphi_{k'}$$

a space average — removes one sum

In 1D, for a discrete system of length L:

$$k=\frac{2\pi n}{L}, \quad n = \text{integer}$$

Δk = spacing of k = $2\pi/L$

$\hookrightarrow \frac{\Delta k L}{2\pi} = 1$

therefore we can write

$$\sum_k = \underbrace{\sum_k \frac{\Delta k L}{2\pi}}_{\sum_k \Delta k = \int dk}$$

$$\Rightarrow \sum_{k'}\frac{(2\pi)^3}{L^3} \equiv \int d\vec{k}' \quad \text{(in 3D)}$$

$$\dot{W}_p = \frac{iq^2}{m}\int d\vec{v}\underbrace{\int d\vec{k}'\sum_k\delta(\vec{k}+\vec{k}')}e^{-i(\omega_k+\omega_{k'})t}\frac{\vec{k}'\cdot\vec{v}}{\omega_k-\vec{k}\cdot\vec{v}}\vec{k}\frac{\partial}{\partial\vec{v}}f_0\varphi_k\varphi_{k'}$$

when you integrate over the δ-function, all the k' become $-k$. This is because we have $\delta(\vec{k}+\vec{k}')$, this forces $k'=-k$ so that the argument of the δ-function is zero

$$= -i\frac{q^2}{m}\int d\vec{v}\sum_k e^{-i(\omega_k+\omega_{-k})t}\frac{\vec{k}\cdot\vec{v}}{\omega_k-\vec{k}\cdot\vec{v}}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0\varphi_k\varphi_{-k}$$

where

$$\varphi(\vec{x},t) = \varphi_k e^{i\vec{k}\cdot\vec{x}-i\omega_k t} + \varphi_{-k}e^{-i\vec{k}\cdot\vec{x}-i\omega_{-k}t}$$

summed, this must be real (so the imaginary parts must go to zero — $A+A^*=0$)

"Reality Conditions":

$\varphi_{-k} = \varphi_k^*$ — complex conjugate

$\omega_{-k} = -\omega_k^*$

$\Rightarrow \varphi_k\varphi_{-k} = |\varphi_k|^2$

→

$$\dot{W}_p = -i\frac{q^2}{m}\int d\vec{v}\sum_k e^{-i(\omega_k+\omega_{-k})t}\frac{\overbrace{\vec{k}\cdot\vec{v}}^{+\omega_k-\omega_k \text{ to match denominator}}}{\omega_k-\vec{k}\cdot\vec{v}}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0|\varphi_k|^2$$

(as we did for Plasma Dispersion, Lec. #9)

$$= i\frac{q^2}{m}\int d\vec{v}\sum_k \underbrace{e^{-i(\omega_k+\omega_{-k})t}}_{\text{write in terms of complex conjugate}}\frac{\vec{k}\cdot\vec{v}-\omega_k+\omega_k}{\vec{k}\cdot\vec{v}-\omega_k}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0|\varphi_k|^2$$

$$= -i\frac{q^2}{m}\int d\vec{v}\sum_k\frac{\omega_k}{\omega_k-\vec{k}\cdot\vec{v}}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0\, e^{-i(\omega_k-\omega_k^*)t}|\varphi_k|^2$$

Let

imaginary component

$$\omega_k = \omega_R + i\gamma_k$$

real component

Landau damping arising from resonant particles

$$\omega_k+\omega_{-k} = \omega_k-\omega_k^*$$
$$= \omega_R+i\gamma_k-\omega_R+i\gamma_k = 2i\gamma_k$$

$$\dot{W}_p = -i\frac{q^2}{m}\int d\vec{v}\sum_k\frac{\omega_k}{\omega_k-\vec{k}\cdot\vec{v}}\vec{k}\cdot\frac{\partial}{\partial\vec{v}}f_0\,\underbrace{e^{\overbrace{-i(2i\gamma_k)t}^{=2\gamma_k t}}|\varphi_k|^2}_{e^{2\gamma_k t}|\varphi_k|^2=\frac{|E_k(t)|^2}{k^2}}$$

where before $\hat{E}=-ik\hat{\varphi}$, with $e^{2\gamma_k t}$ pulled into the definition of $|\hat{E}|^2$

$$\Rightarrow \dot{W}_p = -i\frac{q^2}{m}\sum_k\omega_k\int d^3v\frac{(\vec{k}\cdot\partial/\partial\vec{v}f_0)}{\omega_k-\vec{k}\cdot\vec{v}}\frac{|E_k(t)|^2}{k^2}$$

Recall from Lecture #9, with Linear Theory:

$$\begin{cases}\varepsilon(\omega,\vec{k}) = \dfrac{k^2}{\varphi_{1k\omega}}\left[-4\pi e i\displaystyle\int d\vec{v}\frac{f_{1k\omega}(t=0)}{\omega-\vec{k}\cdot\vec{v}}\right]\\[2ex] \varepsilon(k,\omega) = 1+\dfrac{\omega_{pe}^2}{k^2 n_0}\displaystyle\int d\vec{v}\frac{\vec{k}\cdot\partial/\partial\vec{v}f_0}{\omega-\vec{k}\cdot\vec{v}} = 1+4\pi\chi_e\end{cases}$$

Here,

$$\chi = \int d^3v\frac{\vec{k}\cdot\partial/\partial\vec{v}f_0}{\omega_k-\vec{k}\cdot\vec{v}}\frac{q^2}{m}\frac{1}{k^2}$$

such that

sum is over $\pm k$

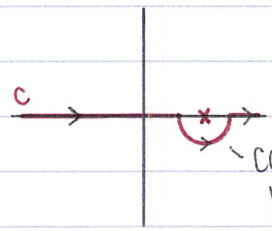
$$\Rightarrow \dot{W}_p = -i\sum_k \omega_k \chi_k |E_k(t)|^2$$

for $+k$:

$$\chi_e(k) = \int d\vec{v}\, \frac{\vec{k}\cdot \partial/\partial\vec{v}\, f_0}{\omega_k - \vec{k}\cdot\vec{v}}\, \frac{q^2}{mk^2}$$

$$= \int d\vec{v}_\perp \int dv_\parallel \frac{k\, \partial/\partial v_\parallel f_0}{\omega_k - k v_\parallel}\, \frac{q^2}{mk^2}$$

$$= -\int d\vec{v}_\perp dv_\parallel \frac{\partial/\partial v_\parallel f_0}{v_\parallel - \omega_k/k}\, \frac{q^2}{mk^2}$$



contour deformed below pole

Residue = $i\pi$

(found in Lecture #9)

→ Inspect imaginary component of linearity term

$$\mathrm{Im}\{\chi_e(k)\} = -\int d\vec{v}_\perp \frac{q^2}{mk^2}\frac{\partial f_0}{\partial v_\parallel}\Big|_{v_p=\omega_k/k}(\pi) + \frac{\partial \chi_e}{\partial \omega}\Big|_{\omega_R}\gamma_k$$

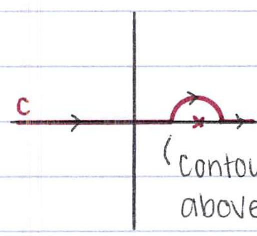
= Im{residue}, residue = $i\pi$

for $-k$:

$$\chi_e(-k) = -\int d\vec{v}_\perp dv_\parallel \frac{\partial/\partial v_\parallel f_0}{v_\parallel + \omega_{-k}/k}\, \frac{q^2}{mk^2}$$

$\omega_{-k} = -\omega_k^*$

$$= -\int d\vec{v}_\perp dv_\parallel \frac{\partial/\partial v_\parallel f_0}{v_\parallel - \omega_k^*/k}\, \frac{q^2}{mk^2}$$



contour deformed above pole

Residue = $-i\pi$

$$\mathrm{Im}\{\chi_e(-k)\} = -\int d\vec{v}_\perp \frac{q^2}{mk^2}\frac{\partial f_0}{\partial v_\parallel}\Big|_{v_p}(-\pi) + \frac{\partial \chi_e}{\partial \omega}\Big|_{\omega_R}(-\gamma_k)$$

= Im{residue}, residue = $-i\pi$

$$= -\mathrm{Im}\{\chi_e(k)\}$$

$$\left.\begin{aligned} \mathrm{Re}\{\chi_e(-k)\} &= \mathrm{Re}\{\chi_e(k)\} \\ \mathrm{Im}\{\chi_e(-k)\} &= -\mathrm{Im}\{\chi_e(k)\} \end{aligned}\right\} \rightarrow \chi_e(-k) = \chi_e^*(k)$$

$\sum_k \omega_k \chi_k$:

Our "reality conditions" must still be true

$$\begin{cases} \omega_k + \omega_{-k} = \omega_k - \omega_k^* \,; & \omega_k = \omega_R + i\gamma_k \\ \chi_k + \chi_{-k} = \chi_k + \chi_k^* \,; & \chi_k = \chi_R + i\chi_I \end{cases}$$

$$\dot{W}_p = -i(\omega_k \chi_k + \omega_{-k}\chi_{-k})|E_k(t)|^2$$

$\rightarrow \omega_k \chi_k = (\omega_R + i\gamma_k)(\chi_R + i\chi_I)$

$= \cancel{\omega_R \chi_R} + i\gamma_k \chi_R + i\omega_R \chi_I - \cancel{\gamma_k \chi_I}$

$\rightarrow \omega_{-k}\chi_{-k} = -\omega_k^* \chi_k^* = -(\omega_R - i\gamma_k)(\chi_R - i\chi_I)$

$= \underbrace{-\cancel{\omega_R \chi_R} + i\gamma_k \chi_R + i\omega_R \chi_I + \cancel{\gamma_k \chi_I}}_{\omega_R \chi_R \text{ \& } \gamma_k \chi_I \text{ cancel out}}$

$$\omega_k \chi_k + \omega_{-k}\chi_{-k} = 2i\gamma_k \chi_R + 2i\omega_R \chi_I$$

$\Rightarrow$ Only the imaginary part $\sum_k \omega_k \chi_k$ survives!

↓ we must extract it ↓

$$\mathrm{Im}\{\omega_k, \chi_k\} \simeq \mathrm{Im}\left\{ \frac{\partial(\omega_k, \chi)}{\partial \omega}\bigg|_{\omega_R} i\gamma_k + \omega_R i \chi_I \right\}$$

$$\dot{W}_p = \sum_k \left[\frac{\partial}{\partial \omega_k}(\omega_k, \chi_e)\bigg|_{\omega_k}(2\gamma_k) + 2\omega_R \chi_I \right] \frac{|E_k|^2}{2}$$

We now need to include the electric field energy

rate of change of electric field energy

$$\dot{W}_E = 2\gamma_k \frac{|E_k|^2}{8\pi} = 2\gamma_k \frac{|E_k|^2}{8\pi} \frac{\partial}{\partial \omega_k}\omega_k$$

Putting this all together...

$$\dot{W} = \dot{W}_p + \dot{W}_E$$

$$= \sum_k \left\{ \left[\frac{\partial}{\partial \omega_k}(\omega_k \chi_k)\bigg|_{\omega_k}(2\gamma_k) + 2\omega_R \chi_I \right] \frac{|E_k|^2}{2} + \left[2\gamma_k \frac{|E_k|^2}{8\pi} \frac{\partial}{\partial \omega_k}\omega_k \right] \right\}$$

$$= \sum_k \left[\frac{\partial}{\partial \omega_k}\Big(\omega_k \underbrace{(1 + 4\pi\chi_k)}_{\varepsilon = 1 + 4\pi\chi}\Big) 2\gamma_k + 2\omega_R(1 + 4\pi\chi_I) \right] \frac{|E_k|^2}{8\pi}$$

$$\Rightarrow \dot{W} = \sum_k \left[\underbrace{\frac{\partial}{\partial \omega_k}(\omega_k \varepsilon_e) 2\gamma_k}_{\substack{\text{change of} \\ \text{wave energy,} \\ \text{"sloshing particles"}}} + \underbrace{2\omega_R \varepsilon_I}_{\text{change of resonant particle energy}} \right] \frac{|E_k|^2}{8\pi}$$

(drop this for determining Wwave)

*The sloshing particles are those that see the wave with a non-zero frequency. Just like in a spring-mass system, the particles oscillating in a wave have kinetic energy

NOW: Find the wave energy

$$\sum_k \frac{\partial}{\partial \omega_k}(\omega_k \varepsilon) 2\gamma_k \frac{|E_k|^2}{8\pi} = \sum_k \frac{\partial}{\partial \omega_k}(\omega_k \varepsilon) \frac{\partial}{\partial t} \frac{|E_k|^2}{8\pi}$$

Proof:

$$|E_k|^2 = k^2 e^{2\gamma_k t} |\varphi_k|^2$$

$$\frac{\partial}{\partial t}|E_k|^2 = k^2 \underbrace{\left(\frac{\partial}{\partial t} e^{2\gamma_k t}\right)}_{2\gamma_k e^{2\gamma_k t}} |\varphi_k|^2 + k^2 e^{2\gamma_k t} \underbrace{\left(\frac{\partial}{\partial t}|\varphi_k|^2\right)}_{2\gamma_k |\varphi_k|^2}$$

$$= 2\gamma_k \underbrace{\left(k^2 e^{2\gamma_k t} |\varphi_k|^2\right)}_{|E_k|^2} = 2\gamma_k |E_k|^2$$

$$= \frac{\partial}{\partial t}\left[\sum_k \frac{\partial}{\partial \omega_k}(\omega_k \varepsilon) \frac{|E_k|^2}{8\pi}\right]$$

⇒ Energy Content of a Wave

$$W_{wave} = \sum_k \frac{\partial}{\partial \omega_k}(\omega_k \varepsilon) \frac{|E_k|^2}{8\pi}$$

→ particle sloshing energy plus electric field energy

*does NOT include energy of wave interacting with resonant particles

Ex: Plasma Waves

$\varepsilon = 1 - \omega_{pe}^2/\omega^2$

$$\frac{\partial}{\partial \omega_k}(\omega_k \varepsilon) = \frac{\partial}{\partial \omega_k}\left[\omega_k\left(1 - \frac{\omega_{pe}^2}{\omega_k^2}\right)\right]$$

$$= 1 + \frac{\omega_{pe}^2}{\omega_k^3}\omega_k = 1 + \underbrace{\frac{\omega_{pe}^2}{\omega_k^2}}$$

for plasma waves, $\varepsilon = 0 \rightarrow \omega_{pe}^2 = \omega_k^2$

$$= 1 + \omega_{pe}^2/\omega_{pe}^2 = 2$$

↓ plugging into the energy content eqn ↓

$$W_{wave} = \underbrace{\sum_k \frac{\partial}{\partial \omega_k}(\omega_k \varepsilon)}_{2} \frac{|E_k|^2}{8\pi} = 2\frac{|E_k|^2}{8\pi}$$

⇒ Equally split between electric field and sloshing particles