Lecture 9 - Four-Vectors & One-Forms

09/27/16

Recap: Lorentz transformation, matrix form

$X^{\alpha'} = \Lambda^{\alpha'}_{\beta} X^{\beta}$

$$\Lambda^{\alpha'}_{\beta} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$X^{\alpha'} = \{x^0, x^1, x^2, x^3\} = \{ct, x, y, z\}$

where Λ is defined such that

$\Lambda^{\alpha}_{\gamma'} \Lambda^{\gamma'}_{\beta} = \delta^{\alpha}_{\beta}$

$$\delta^{\alpha}_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Kronecker delta: there should be no change in the unprimed frame by moving into the primed frame and back

Applying this to the invariant spacetime interval:

$ds^2 = c^2 d\tau^2$

$= \underbrace{\eta_{\alpha\beta}}_{\text{invariant metric tensor}} dx^{\alpha} dx^{\beta}$

$$\eta_{\alpha\beta} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

*the signs on this may change – it's just a convention thing

Four-Vectors

$V^{\alpha'} = \Lambda^{\alpha'}_{\beta} V^{\beta}$

anything that satisfies this is a 4-Vector

We can use the metric tensor to contract the 4-Vector with itself

$\vec{V} \cdot \vec{V} = V^{\alpha'} V^{\beta} \eta_{\alpha'\beta}$

4-Velocity: (how we're moving in spacetime)

position vector

$$\vec{U} = \lim_{\underbrace{\delta\tau \to 0}_{\text{proper time interval}}} \frac{\vec{X}(\tau + \delta\tau) - \vec{X}(\tau)}{\delta\tau} = \frac{d\vec{x}}{d\tau}$$

by the definition of a derivative

recall: $d\tau = dt/\gamma$

$\vec{U} = \frac{d\vec{x}}{d\tau} = \gamma \frac{d\vec{x}}{dt}$ – just the standard 3-Vector velocity, $\vec{v}$

We can use $\vec{U}$ to express the invariant spacetime interval

from $ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$

$\hookrightarrow ds^2 = U^\alpha U^\beta \eta_{\alpha\beta} = c^2 \rightarrow$ the invariant quantity here is just a scalar because $\vec{U}$ is over the proper time interval

$\downarrow$ expanding $\downarrow$

$$\eta_{\alpha\beta} U^\alpha U^\beta = (U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2$$

$$= \gamma^2 [c^2 - \underbrace{v_x^2 - v_y^2 - v_z^2}_{\text{3-velocity, } (\vec{v})^2}] = \gamma^2 [\underbrace{c^2 - (\vec{v})^2}_{= c^2/\gamma^2 \text{, from Lorentz transform.}}] = c^2$$

Vectors $\rightarrow$ 4-Vectors (notation):

$\vec{A} \cdot \vec{B} = A^\alpha B^\beta \eta_{\alpha\beta}$

to do this, we need $\downarrow$

Basis Vectors

$$\left\{ \begin{array}{l} \vec{e}_0 = (1,0,0,0) \\ \vec{e}_1 = (0,1,0,0) \\ \vec{e}_2 = (0,0,1,0) \\ \vec{e}_3 = (0,0,0,1) \end{array} \right\}$$

To write vectors as functions of basis vectors:

$$\vec{A} = \underbrace{A^\alpha \vec{e}_\alpha}_{} = A^{\alpha'} \vec{e}_{\alpha'}$$

remember: there is an implicit sum over repeated indices

$\vec{A}$, as the 4-Vector, is independent of our choice of basis vectors.

$\rightarrow \vec{A}$ is a geometric object that does not care how we measure it

$\Rightarrow \vec{e}_\alpha$ and $\vec{e}_{\alpha'}$ must be related by the Lorentz transformation

$\vec{A} = A^\alpha \vec{e}_\alpha$

$\llcorner A^\alpha = \Lambda^\alpha_{\ \beta'} A^{\beta'}$, the Lorentz transformation

$$= \underbrace{\Lambda^\alpha_{\ \beta'} A^{\beta'}}_{} \vec{e}_\alpha = A^{\alpha'} \vec{e}_{\alpha'}$$

we can relabel these indices so long as we do so in pairs

tensor algebra: $\beta' \rightarrow \alpha'$; $\alpha \rightarrow \beta$

(but only on the left side of the equation; allowable because these are all just dummy indices)

$$\Lambda^\alpha_{\ \beta'} A^{\beta'} \vec{e}_\alpha \rightarrow \Lambda^\beta_{\ \alpha'} A^{\alpha'} \vec{e}_\beta$$

$$\vec{A} = \Lambda^\beta_{\ \alpha'} A^{\alpha'} \vec{e}_\beta = A^{\alpha'} \vec{e}_{\alpha'}$$

$\hookleftarrow$ move to other side, collect terms in $A^{\alpha'}$

$$= \underbrace{(\vec{e}_{\alpha'} - \Lambda^\beta_{\ \alpha'} \vec{e}_\beta) A^{\alpha'} = 0}_{\text{must hold for any arbitrary } A^{\alpha'}}$$

$\Rightarrow (\vec{e}_{\alpha'} - \underbrace{\Lambda^{\beta}_{\alpha'} \vec{e}_{\beta}}_{\text{actually } \vec{e}_{\beta}\Lambda^{\beta}_{\alpha'} \text{ by matrix convention}}) = 0$ independent of $\vec{A}$

↳ following from this ↴

$\vec{e}_{\alpha'} = \vec{e}_{\beta} \underbrace{\Lambda^{\beta}_{\alpha'}}_{\text{unprimed to prime frame}}$, the relation of basis vectors through the Lorentz transformation

*Note: Basis vectors transform in the opposite way as regular vector transformations!

↳ this is expected as we need the two Lorentz transformations to cancel into a kronecker delta

<u>One-Forms</u>

Beginning with the gradient

where the gradient is NOT a vector

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x^0}, \frac{\partial\varphi}{\partial x^1}, \dots\right)$$

Define a "naked gradient" using the chain rule

$$\partial\varphi = \frac{\partial\varphi}{\partial x^{\beta}} \partial x^{\beta}$$

↓ differentiate both sides w.r.t. another frame ↓

$$\frac{\partial\varphi}{\partial x^{\alpha'}} = \frac{\partial\varphi}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x^{\alpha'}}$$

└ we know X must satisfy $x^{\beta} = \Lambda^{\beta}_{\gamma'} x^{\gamma'}$

$$\rightarrow \frac{\partial x^{\beta}}{\partial x^{\alpha'}} = \Lambda^{\beta}_{\gamma'} \underbrace{\frac{\partial x^{\gamma'}}{\partial x^{\alpha'}}}_{\delta^{\gamma'}_{\alpha'}}$$

$= \Lambda^{\beta}_{\alpha'}$ - kronecker delta enabled change in index

Applying this:

$\frac{\partial\varphi}{\partial x^{\alpha'}} = \frac{\partial\varphi}{\partial x^{\beta}} \Lambda^{\beta}_{\alpha'}$ ← where this $\Lambda^{\beta}_{\alpha'}$ takes a primed 4-vector → unprimed frame, or opposite for basis vectors

$\Rightarrow \partial\varphi/\partial x^{\beta}$ behaves as a basis vector!

$= A_{\alpha'}$ *one-forms are covariant components such that they satisfy the form $p_{\alpha'} = p_{\beta}\Lambda^{\beta}_{\alpha'}$

Aside: Contravariant vs. Covariant Components

- 4-Vectors are contravariant components (like kets)
- Basis vectors and one-forms are covariant components (like bras)

"up notation" means contravariant – "up against"

$\vec{A} = A^{\alpha}\vec{e}_{\alpha}$

"down notation" means covariant – "down with"

→ basis vectors and one-forms can be thought of as geometric objects since they are covariant

Basis One-Forms

Contravariant components

$\vec{A} = A^{\alpha}\underbrace{\vec{e}_{\alpha}}_{\text{a list of basis vectors}} = A_{\alpha}\underbrace{\tilde{e}^{\alpha}}_{\text{a list of one-forms}}$

We can dot these two forms together to yield an invariant object

$\vec{A}\cdot\vec{A} = A^{\beta}A_{\alpha}\underbrace{(\vec{e}_{\beta}\cdot\tilde{e}^{\alpha})}_{\text{need change of indices}}$ must $= A^{\alpha}_{\alpha}$

need change of indices $\Rightarrow (\vec{e}_{\beta}\cdot\tilde{e}^{\alpha}) \overset{\text{must}}{=} \delta^{\alpha}_{\beta}$

$= A^{\beta}A_{\alpha}\delta^{\alpha}_{\beta}$

or, more generally, $\eta_{\alpha\beta}$ here

Recap

For some 4-Vector that satisfies

$A^{\alpha'} = \Lambda^{\alpha'}_{\beta}A^{\beta}$, which can be described by a set of basis vectors

contravariant

$\vec{A} = A^{\alpha'}\vec{e}_{\alpha'} \rightarrow \vec{e}_{\alpha'} = \vec{e}_{\beta}\underbrace{\Lambda^{\beta}_{\alpha'}}_{\text{same metric for one-forms}}$ 2

covariant

The same metric can be used to describe the transformation of a related one-form

$p_{\alpha'} = p_{\beta}\Lambda^{\beta}_{\alpha'}$, which can be described by a set of basis one-forms

covariant

$\tilde{\omega}^{\alpha'} = \underbrace{\Lambda^{\alpha'}_{\beta}}_{\text{same metric as for 4-vectors}}\tilde{\omega}^{\beta}$ – contravariant

Such that

$\Rightarrow \vec{A} = A^{\alpha}\vec{e}_{\alpha} = A_{\alpha}\tilde{\omega}^{\alpha}$