Lecture 15 - Curvature

11/01/16

Geodesic: a straight line through curved spacetime

$$\vec{a} = \frac{d\vec{v}}{d\lambda} = 0$$

gradient of the vector along itself

$$= \nabla_{\vec{u}}\vec{u} = \vec{u}\cdot(\nabla\vec{u})$$

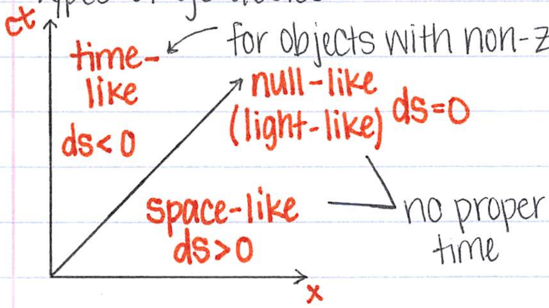
$$= \underbrace{U^{\alpha}{}_{;\beta}}U^{\beta} = 0$$ — no projection along U^{β}

Change in basis vectors within the covariant derivative accounts for curvature to keep tangential components parallel

The projection should never change for a geodesic

↳ No external forces

Types of geodesics:



Curvature

We want to have gravity now, which is just curvature of spacetime

Finding the equations:

- Begin with Newton's gravity

$$\nabla^2\Phi = 4\pi G\rho$$

Want our equation to reduce to this in the weak gravity limit

What is necessary to describe curvature?

- start with the metric at a single point

$$g_{\alpha\beta} \xrightarrow{\text{inertial frame}} \eta_{\alpha\beta}$$

10 independent values (symmetry)

at a single point, you can't even describe the local neighborhood because you can always shift the point (change your frame) → cannot simultaneously describe situations with and without mass

vs. Coordinate trans. $\frac{\partial x^{\alpha}}{\partial x^{\beta'}}$, 16 independent values

→

- add a derivative

$\times\ \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = \underbrace{g_{\alpha\beta,\gamma}}$

we can always force the local area to be flat, $g_{\alpha\beta;\gamma} = 0$

There are frames in which this can tell you nothing (when flat), so you cannot guarantee information on gravity with this

- introduce Christoffel symbols

$\times\ \Gamma^\alpha_{\beta\gamma} = 0$, as implied by the Levi-Civita equation

↳ equivalent to ↴

$\underbrace{A^\alpha = 0}_{} = \frac{du^\alpha}{d\tau}$ — locally inertial / geodesic coordinates

This still lacks information!

(not generally covariant)

Must introduce 2nd derivatives!

Simplest 2nd derivative, covariant form ↴

$\nabla_\gamma \nabla_\beta V_\alpha = V_{\alpha;\beta\gamma}$

↑ generic one-form

This is a tensor (derivatives are covariant) which contains second derivatives of the Christoffel symbols.

→ Expand the covariant derivative w.r.t. γ

$V_{\alpha;\beta\gamma} = [V_{\alpha;\beta}]_{;\gamma}$

$= V_{\alpha;\beta,\gamma} - \underbrace{\Gamma^\sigma_{\alpha\gamma} V_{\sigma;\beta}} - \underbrace{\Gamma^\sigma_{\beta\gamma} V_{\alpha;\sigma}}$

one term for each component, α and β

Each of the three covariant derivatives can be expanded

↳ answer of the form ↴

a one-form

$V_{\alpha;\beta\gamma} = [\ldots] V_{\mu,\beta\gamma} + [\ldots] V_{\rho,\sigma} + [\ldots] V_\rho$

While the sum is a tensor, we cannot say anything about the nature of these components

Where the terms in brackets involve 2nd derivatives in g

* Need an expression involving V_ρ alone!

Construct something of the following form

$$[\nabla_\gamma, \nabla_\beta] V_\alpha = V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta}$$

↑ derivatives in V will cancel out

- consider the lefthand derivative first

$$V_{\alpha;\beta\gamma} = [V_{\alpha;\beta}]_{;\gamma} = V_{\alpha;\beta,\gamma} - \Gamma^\sigma_{\alpha\gamma} V_{\sigma;\beta} - \Gamma^\sigma_{\beta\gamma} V_{\alpha;\sigma}$$

↓ expand covariant derivatives ↓

$$= (V_{\alpha,\gamma} - \Gamma^\sigma_{\alpha\beta} V_\sigma)_{,\gamma} - \Gamma^\sigma_{\alpha\gamma}(V_{\sigma,\beta} - \Gamma^\rho_{\sigma\beta} V_\rho) - \Gamma^\sigma_{\beta\gamma}(V_{\alpha,\sigma} - \Gamma^\rho_{\alpha\sigma} V_\rho)$$

- then similarly for the righthand derivative

$$V_{\alpha;\gamma\beta} = [V_{\alpha;\gamma}]_{;\beta} = V_{\alpha;\gamma,\beta} - \Gamma^\sigma_{\alpha\beta} V_{\sigma;\gamma} - \Gamma^\sigma_{\gamma\beta} V_{\alpha;\sigma}$$

$$= (V_{\alpha,\gamma} - \Gamma^\sigma_{\alpha\gamma} V_\sigma)_{,\beta} - \Gamma^\sigma_{\alpha\beta}(V_{\sigma,\gamma} - \Gamma^\rho_{\sigma\gamma} V_\rho) - \Gamma^\sigma_{\gamma\beta}(V_{\alpha,\sigma} - \Gamma^\rho_{\alpha\sigma} V_\rho)$$

↓ plugging in ↓

$$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = (V_{\alpha,\beta} - \Gamma^\sigma_{\alpha\beta} V_\sigma)_{,\gamma} - (V_{\alpha,\gamma} - \Gamma^\sigma_{\alpha\gamma} V_\sigma)_{,\beta} + \Gamma^\sigma_{\alpha\gamma}(V_{\sigma,\beta} - \Gamma^\rho_{\sigma\beta} V_\rho)$$
$$+ \Gamma^\sigma_{\alpha\beta}(V_{\sigma;\gamma} - \Gamma^\rho_{\sigma\gamma} V_\rho) - \Gamma^\sigma_{\beta\gamma}(V_{\alpha,\sigma} - \Gamma^\rho_{\alpha\sigma} V_\rho) + \Gamma^\sigma_{\gamma\beta}(V_{\alpha,\sigma} - \Gamma^\rho_{\alpha\sigma} V_\rho)$$

$V_{\alpha,\beta,\gamma} = V_{\alpha,\gamma,\beta}$ by commutativity of partial differentiation

$$= \cancel{(V_{\alpha,\beta})_{,\gamma}} - (\Gamma^\sigma_{\alpha\beta})_{,\gamma} V_\sigma - \cancel{\Gamma^\sigma_{\alpha\beta} V_{\sigma;\gamma}} - \cancel{(V_{\alpha,\gamma})_{,\beta}} + (\Gamma^\sigma_{\alpha\gamma})_{,\beta} V_\sigma$$
$$+ \cancel{\Gamma^\sigma_{\alpha\gamma} V_{\sigma,\beta}} - \cancel{\Gamma^\sigma_{\alpha\gamma} V_{\sigma,\beta}} + \Gamma^\sigma_{\alpha\gamma}\Gamma^\rho_{\sigma\beta} V_\rho + \cancel{\Gamma^\sigma_{\alpha\beta} V_{\sigma;\gamma}} - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\gamma} V_\rho$$
$$- \cancel{\Gamma^\sigma_{\beta\gamma} V_{\alpha,\sigma}} + \cancel{\Gamma^\sigma_{\beta\gamma}\Gamma^\rho_{\alpha\sigma} V_\rho} + \cancel{\Gamma^\sigma_{\gamma\beta} V_{\alpha,\sigma}} - \cancel{\Gamma^\sigma_{\gamma\beta}\Gamma^\rho_{\alpha\sigma} V_\rho}$$

$\Gamma^\sigma_{\beta\gamma} = \Gamma^\sigma_{\gamma\beta}$, by symmetry of the connection coefficients

$$= \underbrace{(\Gamma^\sigma_{\alpha\gamma})_{,\beta} V_\sigma - (\Gamma^\sigma_{\alpha\beta})_{,\gamma} V_\sigma} + \Gamma^\sigma_{\alpha\gamma}\Gamma^\rho_{\sigma\beta} V_\rho - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\gamma} V_\rho$$

because these are just arbitrary indices, we can switch $\sigma \to \rho$ in these terms

$$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = (\Gamma^\rho_{\alpha\gamma,\beta} - \Gamma^\rho_{\alpha\beta,\gamma} + \Gamma^\sigma_{\alpha\gamma}\Gamma^\rho_{\sigma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\gamma}) V_\rho$$

↑ since the left-hand-side is a tensor, the term in the parentheses must also be a tensor

Then, knowing

$$[\nabla_\gamma, \nabla_\beta] V_\alpha = V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = R^\rho_{\alpha\beta\gamma} V_\rho$$

We can say the tensor value in parentheses above must correspond to the <u>Riemann Curvature Tensor</u>

$$\Rightarrow R^\rho_{\alpha\beta\gamma} = \Gamma^\rho_{\alpha\gamma,\beta} - \Gamma^\rho_{\alpha\beta,\gamma} + \Gamma^\sigma_{\alpha\gamma}\Gamma^\rho_{\sigma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\gamma}$$

└ potentially 256 independent components

*NOTE: The equation for the Levi-Civita connection means that $R^\rho_{\alpha\beta\gamma}$ contains the metric and its first derivatives

$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} (g_{\alpha\delta,\beta} + g_{\delta\beta,\alpha} - g_{\alpha\beta,\delta})$

↳ The appearance of derivatives of the connection means that $R^{\rho}_{\alpha\beta\gamma}$ also contains second derivatives of the metric

→ In freely-falling frames, the first derivatives of the metric are zero, but the second derivatives (representing tidal forces) do not vanish in general, so the curvature tensor and gravity cannot always be "transformed away"

*there is no convention for the sign of the Riemann tensor, so you may see differences of sign in books

What if we have flat spacetime?

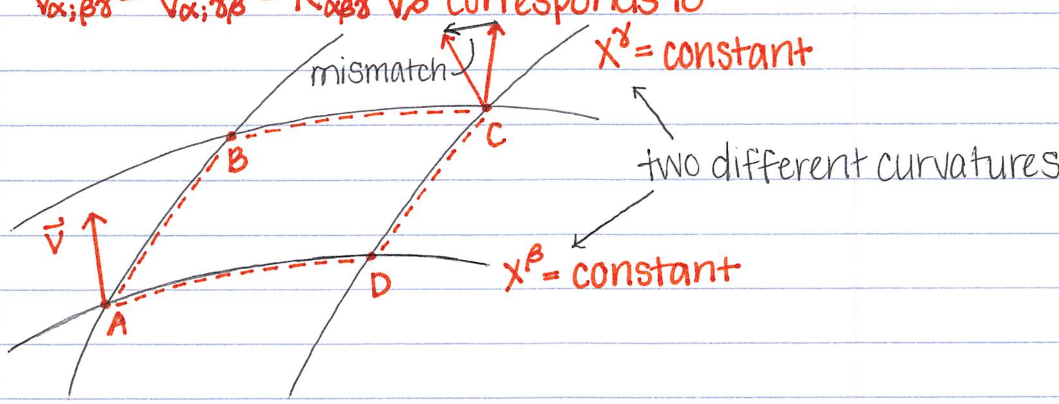
$\Gamma = 0$, $\partial\Gamma = 0$

↳ $R^{\rho}_{\alpha\beta\gamma} = 0$, all structure is lost (as expected for simplification)

i.e., covariant differentiation is commutative in flat space

Geometrically:

$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = R^{\rho}_{\alpha\beta\gamma} V_\rho$ corresponds to



- Vector $\vec{V}$ is first transported A→D→C
  - ↳ associated with $V^{\alpha}{}_{;\beta\gamma}$
- The same vector is then taken A→B→C
  - ↳ associated with $V^{\alpha}{}_{;\gamma\beta}$

*Curvature causes the vectors at C to differ!

A vector parallel transported two ways around the same loop does not match up at the end if there is curvature.

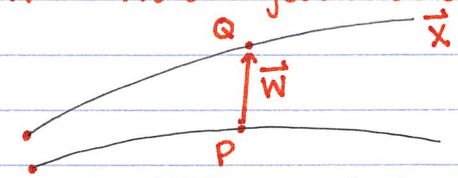
Recall: It is the change in basis vectors in the covariant derivative that allows for parallel transport over a curved surface

↳ on the surface of a sphere, you WILL change orientation

*By parallel transporting to every point on a curvature, one could construct the Riemann Curvature Tensor

Geodesic Deviation:

Two different geodesics can "sense" gravity between them



Two geodesics following curvature – may separate or come together

→ The difference between paths describe the geodesic deviation (a.k.a. the effects of tides)

The equation of geodesic deviation:

Consider the relative distance $\vec{w}$ between two nearby particles at P and Q undergoing geodesic motion (free-fall)

total deriv. $\quad\quad$ $\dot{x}^\gamma = dx^\gamma/d\lambda$, etc.

$$\frac{D^2 w^\alpha}{D\lambda^2} + \underbrace{R^\alpha{}_{\gamma\beta\delta}\,\dot{x}^\gamma \dot{x}^\delta w^\beta}_{\text{tidal acceleration term}} = 0$$

*a tensor equation

where the total/absolute derivatives allow for variations in components caused purely by curved coordinates.

↳ such that $\frac{D^2 w^\alpha}{D\lambda^2} = 0$ in the absence of gravity

The tidal acceleration term therefore represents the effect of gravity that is NOT removed by free-fall (can't be "transformed away")

– In Newtonian physics: tides are caused by a variation in the gravitational field, ∇g, and since $g = -\nabla\phi$, tides are related to $\underbrace{\nabla^2\phi}$

another indication of the connection between curvature and the left-hand-side of $\nabla^2\phi = 4\pi G\rho$

Symmetries Within $R^\rho_{\alpha\beta\gamma}$:

Recall: The fully covariant Riemann Curvature Tensor components are:

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} R^\rho_{\beta\gamma\delta} = g_{\alpha\rho}\left(\Gamma^\rho_{\beta\delta,\gamma} - \Gamma^\rho_{\beta\gamma,\delta} + \Gamma^\sigma_{\beta\delta}\Gamma^\rho_{\sigma\gamma} - \Gamma^\sigma_{\beta\gamma}\Gamma^\rho_{\sigma\delta}\right)$$

→ Specialize to geodesic coordinates in which the connection (but not in general its derivatives) vanish

└ i.e., work in a local Lorentz frame at some point P, such that P is associated with the coordinates $(0, \dots 0)$ and $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $g_{\alpha\beta} = 0$ at P. In the local Lorentz frame, the Christoffel symbols Γ^σ then vanish at P

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho}\left(\Gamma^\rho_{\beta\delta,\gamma} - \Gamma^\rho_{\beta\gamma,\delta}\right)$$

substitute the Levi-Civita equation

↓ $\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}\left(g_{\alpha\delta,\beta} + g_{\delta\beta,\alpha} - g_{\alpha\beta,\delta}\right)$ ↓

$$R_{\alpha\beta\gamma\delta} = \cancel{g_{\alpha\rho}}\left[\tfrac{1}{2}\cancel{g^{\rho\alpha}}\left(g_{\beta\alpha,\delta} + g_{\alpha\delta,\beta} - g_{\beta\delta,\alpha}\right)_{,\gamma} - \tfrac{1}{2}\cancel{g^{\rho\alpha}}\left(g_{\beta\alpha,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right)_{,\delta}\right]$$

$$= \tfrac{1}{2}\left(\underbrace{\cancel{g_{\beta\alpha,\delta\gamma}}} + g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} - \cancel{g_{\beta\alpha,\gamma\delta}} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta}\right)$$

$= g_{\beta\alpha,\gamma\delta}$ by commutativity of partial differentiation

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\left(g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma}\right)$$

↑ the right-hand-side is no longer a tensor, but it allows us to establish symmetries that are tensorial and therefore hold in all frames

The following symmetries are easily established:

$$\left.\begin{aligned} R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta} \\ &= -R_{\beta\alpha\gamma\delta} \\ &= -R_{\alpha\beta\delta\gamma} \end{aligned}\right\} \quad R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$$

*symmetric under exchange of covariant derivatives

→ From these symmetries we can recognize that there are NOT 256 independent components, but only 20!

↳ this corresponds to only one independent contraction

$$R_{\beta\gamma} = g^{\alpha\delta} R_{\alpha\beta\gamma\delta}$$

↓ equivalent notation ↓

$$R_{\alpha\beta} = R^\rho_{\alpha\beta\rho} = g^{\rho\sigma} R_{\sigma\alpha\beta\rho}$$ **The Ricci Tensor**

All other contractions of the Riemann Curvature Tensor are zero due to dependencies.

$$R^\rho_{\rho\alpha\beta} = g^{\rho\sigma} R_{\sigma\rho\alpha\beta} = 0, \text{ etc.}$$

The Ricci Tensor can become the Ricci Scalar by contracting again
↳ $R = g^{\alpha\beta} R_{\alpha\beta} = R^{\alpha}_{\alpha}$

From all this one can construct the Bianchi Identity:
- start with the covariant derivative of the Riemann tensor

$$R_{\alpha\beta\gamma\delta;\mu} = \frac{1}{2}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial^2 g_{\alpha\gamma}}{\partial x^{\delta}\partial x^{\beta}} - \frac{\partial^2 g_{\beta\gamma}}{\partial x^{\delta}\partial x^{\alpha}} - \frac{\partial^2 g_{\alpha\delta}}{\partial x^{\beta}\partial x^{\gamma}} + \frac{\partial^2 g_{\beta\delta}}{\partial x^{\gamma}\partial x^{\alpha}}\right)$$

↓ permute γ, δ, and μ ↓

$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma} = 0$ **The Bianchi Identity**

└ where differentiating the Riemann tensor yields the value of geodesic coordinates

NOTE: this identity is important in calculating the divergence of the Ricci tensor (needed for Einstein Field Equations)

As with the Ricci tensor, you can also contract the Bianchi Identity

$g^{\alpha\delta}(R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma})$

$R_{\beta\gamma;\mu} + R^{\delta}_{\beta\mu\gamma;\delta} - R_{\beta\mu;\gamma} = 0$ **The Contracted Bianchi Identity**

↓ contract β and γ ↓

$g^{\beta\gamma}(R_{\beta\gamma;\mu} + R^{\delta}_{\beta\mu\gamma;\delta} - R_{\beta\mu;\gamma}) = 0$

$R_{;\mu} - R^{\delta}_{\mu;\delta} - R^{\gamma}_{\mu;\gamma} = 0$

↳ change $\delta \to \alpha$

$R_{;\mu} - R^{\alpha}_{\mu;\alpha} - R^{\gamma}_{\mu;\gamma} = 0$

↓ contract μ and β ↓

$g^{\mu\beta}(R_{;\mu} - R^{\alpha}_{\mu;\alpha} - R^{\gamma}_{\mu;\gamma}) = 0$

$R^{\mu\beta}{}_{;\mu} \underbrace{- R^{\alpha\beta}{}_{;\alpha} - R^{\gamma\beta}{}_{;\gamma}} = 0$

↳ move to other side; $R^{\gamma\beta}{}_{;\gamma} = R^{\alpha\beta}{}_{;\alpha}$ ← because just arbitrary indices

$\underbrace{R^{\mu\beta}{}_{;\mu}}_{= R_{;\mu} g^{\mu\beta}} = 2R^{\alpha\beta}{}_{;\alpha}$

$R^{\alpha\beta}{}_{;\alpha} = \frac{1}{2} R_{;\mu} g^{\mu\beta}$

⇒ this balances the energy-momentum conservation equations $T^{\alpha\beta}{}_{;\alpha} = 0$ (from Lecture #11)

→

In Search of Einstein's Equations

We seek a relativistic version of the Newtonian equation

$$\nabla^2 \Phi = 4\pi G \rho$$

The relativistic analogue of the density, ρ, is the stress-energy tensor, $T^{\alpha\beta}$. Start with this

$\rho \to T^{\alpha\beta}$, which satisfies the covariant derivative

$$T^{\alpha\beta}{}_{;\alpha} = 0$$

Other hints from the Newtonian equation:

- Φ is closely related to the metric
- ∇^2 suggests that we should look for some tensor involving the second derivatives of the metric. $g_{\alpha\beta;\gamma\delta}$
- The metric should be a $\binom{2}{0}$ tensor like $T^{\alpha\beta}$

↳ The contravariant form of the Ricci tensor satisfies these conditions

Ansatz: some constant

$$R^{\alpha\beta} = k T^{\alpha\beta}$$

Note: Both $R^{\alpha\beta}$ and $T^{\alpha\beta}$ are symmetric

↓ take divergence of both sides ↓

$$\underbrace{R^{\alpha\beta}{}_{;\alpha}}_{=\frac{1}{2}R_{,\alpha}g^{\alpha\beta} \neq 0\text{, as presented earlier}} = k \overbrace{T^{\alpha\beta}{}_{;\alpha}}^{=0\text{, by definition}}$$

The divergence of the lefthand-side does not vanish!

$\Rightarrow R^{\alpha\beta} = k T^{\alpha\beta}$ cannot be right!

Make it work:

Define a new tensor, <u>the Einstein Tensor</u>

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}R g^{\alpha\beta}$$

↳ satisfies

$$G^{\alpha\beta}{}_{;\alpha} = \left(R^{\alpha\beta} - \frac{1}{2}R g^{\alpha\beta}\right)_{;\alpha}$$

$R_{;\alpha} = R_{,\alpha}$ by definition of R, the Ricci Scalar

$$= \underbrace{R^{\alpha\beta}{}_{;\alpha}}_{=\frac{1}{2}R_{,\alpha}g^{\alpha\beta}} - \frac{1}{2}\overbrace{R_{;\alpha}}^{R_{;\alpha}=R_{,\alpha}} g^{\alpha\beta} - \frac{1}{2}R\underbrace{g^{\alpha\beta}{}_{;\alpha}}_{\nabla g = 0}$$

$$= 0$$

⇒ Einstein's Field Equations ⇐

$G^{\alpha\beta} = \kappa T^{\alpha\beta}$

$\hookrightarrow R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} = \kappa T^{\alpha\beta}$

*The field equations are second order, non-linear differential equations for the metric

The Newtonian Limit:

Einstein's equations must reduce to $\nabla^2 \Phi = 4\pi G\rho$ in the case of slow motion in weak fields.

Take contraction with $g_{\alpha\beta}$ to change the form

$$g_{\alpha\beta} G^{\alpha\beta} = \underbrace{(g_{\alpha\beta} R^{\alpha\beta})}_{= R \text{, the Ricci scalar}} - \frac{1}{2} R \underbrace{(g_{\alpha\beta} g^{\alpha\beta})}_{= \delta^\alpha_\alpha = 4 \text{, the sum over diagonal terms}} = \kappa \underbrace{(g_{\alpha\beta} T^{\alpha\beta})}_{= T \text{, a scalar value}}$$

$$= R - 2R = \kappa T$$

$$-R = \kappa T$$

$$R^{\alpha\beta} + \frac{1}{2}(-R) g^{\alpha\beta} = \kappa T^{\alpha\beta}$$

plug in found value

$$R^{\alpha\beta} + \frac{1}{2}\kappa T g^{\alpha\beta} = \kappa T^{\alpha\beta}$$

$$R^{\alpha\beta} = \kappa \left(T^{\alpha\beta} - \frac{1}{2} T g^{\alpha\beta}\right)$$

↓ lower indices to covariant form ↓

$$R_{\alpha\beta} = \kappa \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right)$$

$\hookrightarrow T_{\alpha\beta} =$ stress-energy tensor $= \left(\rho + \frac{p}{c^2}\right) U_\alpha U_\beta - p g_{\alpha\beta}$

Required limits:

① $p/c^2 \ll \rho$, the Newtonian limit

pressure suppressed by c^2 very small

$\rightarrow T_{\alpha\beta} \approx \rho U_\alpha U_\beta$

② $g_{\alpha\beta} \approx \eta_{\alpha\beta}$, the weak gravity limit

③ $U^i \ll U^0 \approx c$, the slow motion limit

$\rightarrow T = \rho c^2$

④ solution must be time-independent

⇒ Only T_{00} is non-zero! All spatial terms are negligible under these assumptions

$g_{\alpha\beta} \approx \eta_{\alpha\beta} \rightarrow g_{00} = 1$

$$U_0 = g_{0\alpha} U^\alpha$$
$$\approx g_{00} U^0 \approx c$$
$\hookrightarrow T_{00} = \rho U_0 U_0 \approx \rho c^2$

What about $R_{\alpha\beta}$?

The 00-component of $R_{\alpha\beta}$ is

$$R_{00} = \underbrace{\Gamma^{\rho}_{0\rho,0}}_{=0} - \Gamma^{\rho}_{00,\rho} + \underbrace{\Gamma^{\sigma}_{0\rho}\Gamma^{\rho}_{\sigma 0} - \Gamma^{\sigma}_{00}\Gamma^{\rho}_{\sigma\rho}}_{\text{all } \Gamma \text{ are small, so these must be negligible}}$$

under time-idependence assumption

$$R_{00} \approx \underbrace{-\Gamma^{i}_{00,i}}$$

from Geodesics, we know

$$\Gamma^{i}_{00} = \Phi_{i}/c^2$$

$\downarrow$ plugging in $\downarrow$

$$R_{00} \approx -\left(\frac{\Phi_{,i}}{c^2}\right)_{,i} = -\frac{1}{c^2}\Phi_{,ii}$$

$$= -\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial x^i \partial x^i} = -\frac{1}{c^2}\nabla^2\Phi$$

Substitute into field equations $\downarrow$

$$R_{00} = k\left(T_{00} - \tfrac{1}{2}T g_{00}\right)$$

$$-\frac{1}{c^2}\nabla^2\Phi = k\left(\rho c^2 - \tfrac{1}{2}(\rho c^2)(1)\right)$$

$$\nabla^2\Phi = \underbrace{-\frac{kc^4}{2}}_{\text{must } = +4\pi G}\rho$$

$\rightarrow k = \dfrac{-8\pi G}{c^4}$ yields the Newtonian equation as required!

The full Einstein Field Equations are then

$$\Rightarrow R^{\alpha\beta} - \tfrac{1}{2}R g^{\alpha\beta} = -\frac{8\pi G}{c^4}T^{\alpha\beta} \Leftarrow$$

*Note that 10 independent equations have replaced the singular equation $\nabla^2\Phi = 4\pi G\rho$