Lecture 12 - Curved Spacetime Metrics

10/13/16

For a number flux of particles, $\bar{N}$

$$\vec{\nabla}(\vec{N}) = \frac{\partial N^{\alpha}}{\partial x^{\alpha}} = N^{\alpha}{}_{,\alpha} = 0$$

conservation equations

and for rank-two tensors

$$\partial_{\alpha} T^{\alpha\beta} = T^{\alpha\beta}{}_{,\alpha} = 0$$

where in determining the Stress-Energy Tensor we identified the unknowns:

ρ, T/p, U^{μ}

but this is not really a 4-vector because

$$\vec{U}\cdot\vec{U} = c^2 = U^{\alpha}U_{\alpha} = U^{\alpha}U^{\beta}\eta_{\alpha\beta}$$

stems from the invariant

$$\frac{ds^2}{d\tau^2} = \eta_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}$$

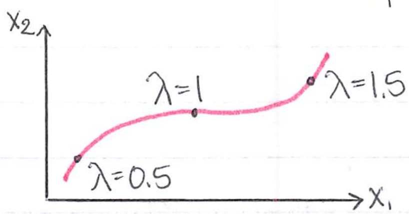
Then, continuing our treatment of gravity from last time:

- straight lines intersecting due to the curvature of spacetime -

What is a curve?

$x^{\alpha} = x^{\alpha}(\lambda)$ parameterized function

$\hookrightarrow x^{\alpha'} = x^{\alpha'}(x^1, x^2, \ldots, x^{\beta}, x^N)$



Can also be a function of a "list" (i.e., another tensor)

$\Rightarrow$ A coordinate transform

e.g. $r = (x^2+y^2)^{1/2}$

$\theta = \arccos(x/r)$

* Coordinate transforms are MORE GENERAL than the Lorentz transformation

Equivalent to write

$$dx^{\alpha'} = \Lambda^{\alpha'}_{\beta}\, dx^{\beta}$$

$$dx^{\alpha} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}}dx^{\beta} = L^{\alpha'}_{\beta}\, dx^{\beta}$$

Lorentz transformation

BUT! The Minkowski metric here in its current form is not general for all coordinate systems!

For a flat Cartesian system:

$ds^2 = \eta_{\gamma\delta}\, dw^\gamma dw^\delta$ (freely falling frame)

└ normal Minkowski

and we know we can write

$$dw^\gamma = \frac{dw^\gamma}{dx^\alpha} dx^\alpha$$ (transition to lab frame)

↓ plugging in ↓

$$ds^2 = \underbrace{\eta_{\gamma\delta} \frac{\partial w^\gamma}{\partial x^\alpha} \frac{\partial w^\delta}{\partial x^\beta}}_{g_{\alpha\beta}} dx^\alpha dx^\beta$$

$g_{\alpha\beta}$ — for this to be invariant, this must be the most general metric you can write on α and β to produce ds^2

using this to rewrite a known quantity

$U^\alpha U^\beta \eta_{\alpha\beta} = c^2$

$= U^\alpha U^\beta g_{\alpha\beta} \Rightarrow$ Anywhere you used η before, you may now use g — it behaves in exactly the same way, including raising and lowering indices.

$$g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$$

On to <u>REAL Gravity</u>

Components of g for different coordinate systems:

① Stationary cartesian invariant interval

$ds^2 = dx^2 + dy^2 + dz^2$

↳ $g_{xx} = g_{yy} = g_{zz} = 1$

② Cylindrical polar invariant interval

$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$

↳ $g_{rr} = 1,\ g_{\varphi\varphi} = r^2,\ g_{zz} = 1$

③ Spherical invariant interval

$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 + dr\, d\theta$

↳ $g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{\varphi\varphi} = r^2\sin^2(\theta),\ g_{r\theta} = \frac{1}{2} = g_{\theta r}$*

↑ $1 \div (2 \text{ components})$ →

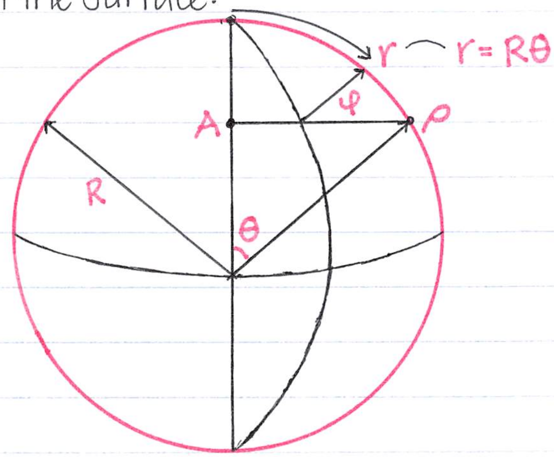
*Note: When coordinates are commutative, the metric is symmetric.

$$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$$
$$= g_{\alpha\beta}\,dx^\beta dx^\alpha$$
$$= g_{\beta\alpha}\,dx^\alpha dx^\beta$$

** the symmetry of $g_{\alpha\beta}$ is non-physical in origin

Creating a Curved Spacetime Metric

on the surface:



Using standard geometry:

$A\rho = R\sin(\theta)$

↳ distance ↴

$ds_\varphi = R\sin(\theta)d\varphi$

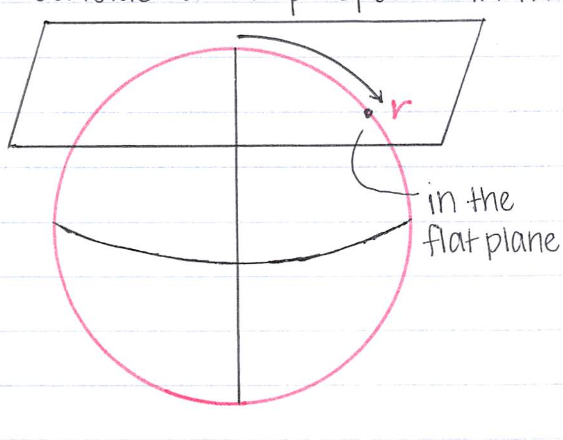
$ds^2 = dr^2 + R^2\sin^2(\theta)\,d\varphi^2$

$= dr^2 + R^2\sin^2(r/R)\,d\varphi^2$

the metric of a 2D space of constant curvature

How does the manifestation of this curvature happen?

↳ consider a flat perspective in the curved frame



I, in flat space, spin in a circle with my arms extended outward.

Are my arms in flat or curved spacetime while spinning?

In curved spacetime

$C = 2\pi R\sin(r/R)$ → For Earth: $r = 10$ km, $R = 6370$ km

which is $< 2\pi r$, the flat spacetime value

$C = 62.831827 < 2\pi r = 62.831853$

⇒ the difference is small, but effects are non-negligible; should treat spinning arms in curved spacetime.

What about in Cosmology?

3D sphere equation:

$$\underbrace{x^2+y^2}_{r^2}+z^2=R^2$$

$r^2 + z^2 = R^2$ (polar coordinates)

Neglecting time, we know this metric to be

$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$, the Euclidean metric

→ We want to restrict our coordinates to the surface of the sphere, not access full 3D-space

take derivative of sphere equation to eliminate dz

$$\partial(r^2+z^2)=\partial(R^2)$$

$$2r\,dr + 2z\,dz = 0$$

↓ plugging in for dz^2 in ds^2 eqn. ↓

$$ds^2 = dr^2 + r^2 d\theta^2 + \frac{r^2 dr^2}{z^2}$$

group terms

plug in for $z^2 = R^2 - r^2$

$$= \frac{dr^2}{1-(r^2/R^2)} + r^2 d\theta^2$$

→ define $k \equiv$ curvature $\equiv 1/R^2$

$$ds^2 = \frac{dr^2}{1-kr^2} + r^2 d\theta^2$$

A metric of 2D-space of constant curvature

⇒ $k=0$ yields the flat spacetime solution

- $k>0$ can be "embedded" in 3D as the surface of a sphere
- $k<0$ cannot be embedded, but is still a valid geometry (saddle)

Returning to Tensor Calculus

In Special Relativity we could write (for the Cartesian case)

$$\frac{\partial \vec{V}}{\partial x^\beta} = \left[\frac{\partial (V^\alpha)}{\partial x^\beta}\right]\vec{e}_\alpha$$

because $\vec{e}_\alpha$ does not change in SR

*here we have assumed that there is no spatial dependence on the basis vectors, so they pull out of the derivative

⇒ NOT TRUE IN GENERAL!

Generally,

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + \underbrace{\overbrace{V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}}^{\text{this term previously} = 0}}_{\text{basis vectors now spatially dependent}}$$

Where we may introduce the Christoffel coefficients

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta}\vec{e}_\gamma$$

a general basis vector; uncorrelated to α, β

a sum over all basis vectors (linear combination)

↳ connection/Christoffel coefficients

Plugging in: (with some arbitrary index switching)

$$\frac{\partial \vec{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta}V^\gamma\right)\vec{e}_\alpha$$

pulled out like term; allowable through our use of Γ

⇒ This is NOT fully covariant, but is now the correct derivative for a vector

↳ the "covariant derivative"