12/7/12

Partial Wave Analysis

Recap: One-Dimensional Schrödinger Equation

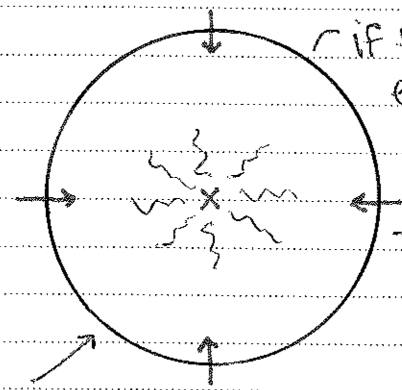
① $\left[\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right] u_\ell(r) = E\, u_\ell(r)$

Boundary condition: $u_\ell(r=0) = 0$

② $u_\ell(r) \xrightarrow[r\to\infty]{} A\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$ phase shift

③ $\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left|\sum_{\ell=0}^{\infty} \underbrace{\frac{(2\ell+1)}{k}\left(e^{i2\delta_\ell} - 1\right)}_{f_\ell} P_\ell(\cos(\theta))\right|^2$

$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell)$



incoming/outgoing spherical wave

- if there is no potential, wavefunction will evolve in time like it shrinks and then expands (free particle)
- if there is a potential, complicated things may happen, but at infinity the wavefunction will look like the free particle because of unitarity, but it will have picked up some phase shift, $\delta_\ell(k)$

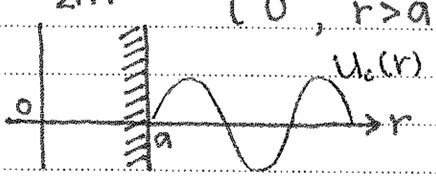
* note that this is still the TISE!

But our initial condition is a plane wave...

→ a plane wave is just a linear combination of spherical waves, so instead of a single 'ℓ', we now have all 'ℓ's' (different partial waves will have different phases)

Example: Hard Sphere (S-wave ⇒ $\ell = 0$)

$\frac{-\hbar^2}{2m}u_0'' + \begin{cases}\infty, & r<a \\ 0, & r>a\end{cases}, \quad u_0(r) = E_0 u(r)$

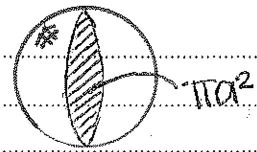


⇒ $u_0(r) = A\sin(kr + \delta_0)$

Boundary condition: $u_0(r=a) = 0$

⇒ $ka + \delta_0 = 0$

$\delta_0 = -ka$



$$\sigma\Big|_{s\text{-wave}} = \frac{4\pi}{k^2}\sin^2(ka) \xrightarrow[ka\ll 1]{} 4\pi a^2$$

└ 4x the geometric cross-section of the sphere

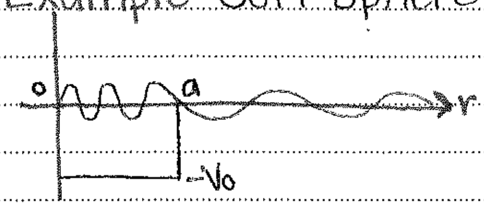
└ small energy approximation

If we were to go to higher ℓ-values, we would need SBF's

Things to know about spherical bessel functions:

1. asymtotic behavior
2. where the zeroes are

Example: Soft Sphere (S-wave)



$$u(r) = \begin{cases} A\sin(\kappa r), & r<a \\ B\sin(kr+\delta), & r>a \end{cases}$$

$\kappa = \frac{1}{\hbar}\sqrt{2m(E+V_0)}$

$k = \frac{1}{\hbar}\sqrt{2mE}$

Boundary condition: $u(a_+) = u(a_-)$

$\rightarrow A\sin(\kappa a) = B\sin(ka+\delta)$

Boundary condition: $u'(a_+) = u'(a_-)$

$\rightarrow A\kappa\cos(\kappa a) = Bk\cos(ka+\delta)$

└ although there are many unknowns, we only need to find this (normalization arbitrary)

$$\Rightarrow \tan(\delta) = \frac{\frac{k}{\kappa}\tan(\kappa a) - \tan(ka)}{1+\frac{k}{\kappa}\tan(\kappa a)\tan(ka)}$$

$$\xrightarrow[ka\ll 1]{} \delta \simeq \tan(\delta) \simeq ka\left[\frac{\tan(\kappa a)}{\kappa a} - 1\right]$$

There's an energy $\underbrace{\overset{\tan\to\infty}{\kappa a = \pi/2}}_{\text{bound states appear}} \Rightarrow \delta = \pi/2$

↑ largest cross section possible! $\sin^2(\pi/2) = 1$

→ You reach the unitarity bound!