12/5/12

Born Approximation, con't

Scattering amplitude:

$$\underbrace{f(\theta,\varphi)}_{\text{direction of } \mathbb{k}-\mathbb{k}_0 \text{ fixed}} = \frac{-m}{2\pi\hbar^2}\int d^3\mathbb{r}'\, V(\mathbb{r}')\, e^{-i\overbrace{(\mathbb{k}-\mathbb{k}_0)}^{\mathbb{q}\text{ - momentum transfer}}\mathbb{r}'}$$

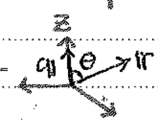
$\mathbb{k}_0$ — in the direction of the incident wave

$\mathbb{k}$ — in the direction of the reflected wave

Simplification:

If $V(\mathbb{r}) = V(r)$ (spherically symmetric)

$$f = \frac{-m}{2\pi\hbar^2}\int_0^r dr' \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_{-1}^{1} d\cos(\theta)\, e^{-iqr'\cos(\theta)}\, r'^2\, V(r')$$



$$= \frac{-m}{\hbar^2}\int_0^\infty dr'\, r'^2\, V(r')\, \underbrace{\left.\frac{e^{iqr'\cos(\theta)}}{-iqr'}\right|_{-1}^{1}}_{\frac{e^{-iqr'}-e^{iqr}}{-iqr'} = \frac{2}{qr'}\sin(qr')}$$

$$= \frac{-2m}{q\hbar^2}\int_0^\infty dr'\, r'\, V(r')\sin(qr')$$

Example: Soft Sphere (sphere has a potential to interact with incoming particles, but particles can also potentially pass through)

$$V(r) = \begin{cases} V_0, & r<R \\ 0, & r>R \end{cases}$$

R — radius of the sphere

$V(r)$ — spherically symmetric

$$f(\theta) = \frac{-2m}{q\hbar^2}\int_0^R dr\, r\, V_0 \sin(qr)$$

$$= \frac{-2m}{q^3\hbar^2} V_0\left[\sin(qR) - qR\cos(qR)\right] \quad \text{(exact solution)}$$

$$\simeq \frac{-2mV_0}{q\hbar^2}\int_0^R dr\, r\, \underbrace{\sin(qr)}_{qr + \frac{(qr)^3}{3!} + \dots}$$

expand for qR very small

$$= \frac{-2mV_0}{q\hbar^2}\left[\frac{qR^2}{2} + \frac{q^3}{3!}\frac{R^5}{5} + \dots\right]$$

$$\simeq \frac{-2mV_0}{\hbar^2}R^2 + \dots$$

for small q (small E), the leading term (scattering amplitude at zero momentum) is independent of q! It's a constant!

→ Even if $V(r)$ is not constant, if it is some complicated thing, this allows us to say that our scattering amplitude is just $f \sim \int_0^\infty dr\, r^2 V(r)$ no matter what is applied (assumed V does not vary too quickly)

Partial Wave Analysis

Solving the 3-dimensional Schrödinger equation with a ψ that isn't necessarily spherically symmetric

$$\psi(r,\theta) = \overbrace{A\sum_{\ell=0}^{\infty} \frac{u_\ell(r)}{r} P_\ell(\cos(\theta))}^{\text{ANY function can be expanded like this}} \quad , \quad u_\ell(0)=0$$

necessary or the ψ blows up

assume no φ-dependence

Legendre Polynomials

different coefficients for each value of r

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + V\right]\psi = E\psi \Rightarrow \left[\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r)\right]u_\ell(r) = E\,u_\ell(r)$$

*this does not apply for the Coulomb force

centrifugal barrier

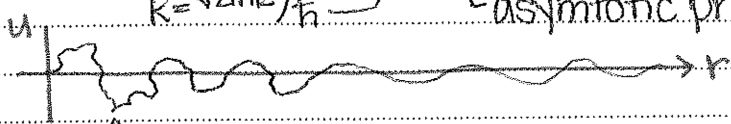
assume potential goes to 0 really quickly for large r (fast = faster than $1/r^2$)

at large enough r: free particle behavior (plane wave)

$$u_\ell(r) \longrightarrow \sin(kr + \bar{\delta}_\ell)$$

$k = \sqrt{2mE}/\hbar$

asymtotic phase, ℓ-dependent



any arbitrary function will, at large r, become a sine wave

$$\psi(r,\theta) = A\sum_{\ell=0}^{\infty} a_\ell \frac{u_\ell(r)}{r} P_\ell(\cos(\theta))$$

$$\xrightarrow[r\to\infty]{} A\sum_{\ell=0}^{\infty} a_\ell \frac{\sin(kr+\bar{\delta}_\ell)}{r} P_\ell(\cos(\theta))$$

Want to choose a_ℓ such that this wavefunction look like a scattering problem at large r (plane wave in → spherical wave out)

Goal: Choose a_ℓ such that $\psi(r,\theta)$ looks like

$$\psi = A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$$

→

Boundary Condition:

$$\psi(r,\theta) \xrightarrow[r\to\infty]{} A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$$

$z = r\cos(\theta)$

$e^{ikz} = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell} j_{\ell}(kr) P_{\ell}(\cos(\theta))$

j_ℓ: spherical bessel functions

$f(\theta) = \sum_{\ell=0}^{\infty} f_\ell P_\ell(\cos(\theta))$

P_ℓ: orthogonal set of polynomials

$$f_\ell = \frac{2\ell+1}{2}\int d\cos(\theta) f(\theta) P_\ell(\cos(\theta))$$

$$= A\sum_{\ell=0}^{\infty}\left[(2\ell+1)i^{\ell} j_\ell(kr) + f_\ell \frac{e^{ikr}}{r}\right] P_\ell(\cos(\theta))$$

dependent on how this behaves at large r

$$\xrightarrow[r\to\infty]{} A\sum_{\ell=0}^{\infty} P_\ell(\cos(\theta))\left[\frac{(2\ell+1)i^{\ell}}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right) + f_\ell\frac{e^{ikr}}{r}\right]$$

general solution for the Schrödinger equation

$$a_\ell \frac{e^{ikr}e^{i\bar{\delta}_\ell} - e^{-ikr}e^{-\bar{\delta}_\ell}}{2ir} = \frac{(2\ell+1)}{kr} e^{i\ell\pi/2}\frac{\left(e^{ikr - i\ell\pi/2} - e^{-ikr + i\ell\pi/2}\right)}{2i} + f_\ell\frac{e^{ikr}}{r}$$

$i^\ell = e^{i\ell\pi/2}$

$$-a_\ell e^{-i\bar{\delta}_\ell} = \frac{-(2\ell+1)}{k} e^{i\ell\pi} \quad (e^{i\pi})^{\ell} = (-1)^{\ell}$$

$$\Rightarrow a_\ell = \frac{(2\ell+1)}{k} e^{i(\bar{\delta}_\ell - \ell\pi)}$$

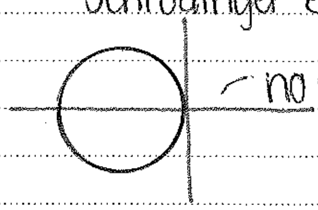
$$a_\ell e^{i\bar{\delta}_\ell} = \frac{(2\ell+1)}{k} + f_\ell$$

$$\Rightarrow f_\ell = \frac{(2\ell+1)}{k} e^{2i\underbrace{(\bar{\delta}_\ell - \ell\pi/2)}_{\delta_\ell = \text{phase shift}}} - \frac{(2\ell+1)}{k}$$

bounded!

$$\Rightarrow f_\ell = \frac{2\ell+1}{k}\left(e^{i2\delta_\ell} - 1\right)$$

partial wave $\Rightarrow$ allows you to solve multiple one-dimensional Schrödinger equations instead of trying to solve the three-dimensional Schrödinger equation for the full wave



no matter how large V is, f is bounded to a circle−1 for a certain energy

$\rightarrow$ Different from Born appx. where V must be small to be applicable total scattering cross-section

$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega\, |f(\theta)|^2$

$= \int d\Omega \sum_{l', l=0}^{\infty} f_l P_l(\cos(\theta)) f_{l'}^* P_{l'}(\cos(\theta))$

$= \sum_{l', l} f_l f_{l'}^* \underbrace{\int d\Omega\, P_l(\cos(\theta)) P_{l'}(\cos(\theta))}_{(4\pi/2l+1)\delta_{ll'}}$ — because of orthogonality

$= \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} |f_l|^2 = \sum_{l=0}^{\infty} \frac{4\pi}{(2l+1)} \frac{(2l+1)^2}{K^2} \underbrace{(1+1+2\cos(2\delta_l))}_{\sin^2(\delta_l)}$

$\sigma = \frac{4\pi}{K^2} \sum_{l=0}^{\infty} (2l+1)\sin^2(\delta_l)$

Unitarity Bound:

if unbounded, violates unitarity

$\sigma_l = \frac{4\pi}{K^2}(2l+1)\sin^2(\delta_l) \leq \frac{4\pi}{K^2}(2l+1)$

partial wave cross-section is bounded

* Unitarity → every particle that goes in comes out (constant magnitude)