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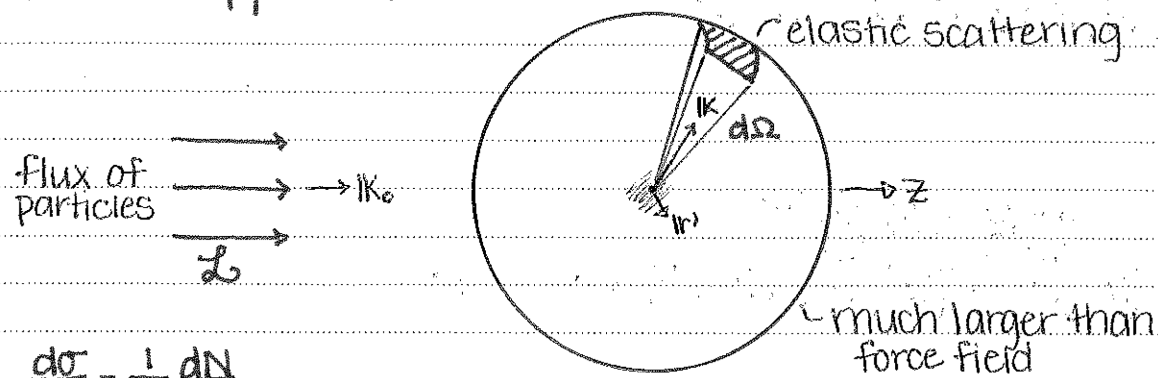
Born Approximation



$$\frac{d\sigma}{d\Omega} = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}$$

differential cross-section: what size hole on the left where the particles originate has the same size as the detection hole on the infinite circle

In QM: Solutions of the TISE that take the form

$$\psi(r,\theta) \underset{r\to\infty}{\sim} A\Big[\underbrace{e^{ikz}}_{\psi_{in}} + \underbrace{f(\theta)\frac{e^{ikr}}{r}}_{\psi_{scat}}\Big]$$

$$\frac{d\sigma}{d\Omega} = |\underbrace{f(\theta,\varphi)}_{\text{scattering amplitude, typically complex}}|^2$$

Born Approximation: $V_{small} \to \psi_{scat} \ll \psi_{in}$

just a plane wave, solution to the free Schrödinger eq.

$$\frac{-\hbar^2}{2m}\nabla_r^2\left(\cancel{\psi_{in}(\mathbb{r})} + \psi_{scat}(\mathbb{r})\right) + \underbrace{V(\mathbb{r})\left(\psi_{in}(\mathbb{r}) + \psi_{scat}(\mathbb{r})\right)}_{V(\mathbb{r})\psi_{in}(\mathbb{r})} = E\left(\cancel{\psi_{in}(\mathbb{r})} + \psi_{scat}(\mathbb{r})\right)$$

$$\left[\nabla_r^2 + \frac{2mE}{\hbar^2}\right]\psi_{scat}(\mathbb{r}) = -\left(\frac{2m}{\hbar^2}\right)V(\mathbb{r})\psi_{in}(\mathbb{r})$$

→ Multiply by the Greene's function; the inverse

$$\psi_{scat}(\mathbb{r}) = -\left(\frac{2m}{\hbar^2}\right)\int d^3\mathbb{r}'\, G(\mathbb{r},\mathbb{r}')\, V(\mathbb{r}')\,\psi_{in}(\mathbb{r}')$$

$$[\nabla_r^2 + k^2]\,G(\mathbb{r},\mathbb{r}') = \delta(\mathbb{r}-\mathbb{r}')$$

applicable for any potential

What is $G(\mathbb{r},\mathbb{r}')$?

$$G(\mathbb{r},\mathbb{r}') = \frac{-1}{4\pi}\frac{e^{ik|\mathbb{r}-\mathbb{r}'|}}{|\mathbb{r}-\mathbb{r}'|} \quad (\text{units of } 1/\text{length})$$

$\mathbb{r}'$ is a parameter, not a variable. can choose to center coordinates around $\mathbb{r}'$ ($\mathbb{r}'=0$)

In spherical coordinates centered around $\mathbf{r}'$:

$$\nabla_r^2\left(\frac{e^{ikr}}{r}\right)\left(\frac{-1}{4\pi}\right)=\frac{-1}{4\pi}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\frac{e^{ikr}}{r}$$

$$\frac{d}{dr}\frac{e^{ikr}}{r} = \left[\frac{ik}{r}-\frac{1}{r^2}\right]e^{ikr}$$

$$r^2\frac{d}{dr}\frac{e^{ikr}}{r} = [ikr-1]e^{ikr}$$

$$\frac{d}{dr}r^2\frac{d}{dr}\frac{e^{ikr}}{r} = [\cancel{ik}-k^2r-\cancel{ik}]e^{ikr}$$

$$=-\frac{1}{4\pi}(-k^2)\frac{1}{r}e^{ikr}$$

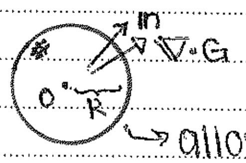
$$\left[\nabla_r^2+k^2\right]\left(\frac{-1}{4\pi}\frac{e^{ikr}}{r}\right)=\frac{-1}{4\pi}\frac{e^{ikr}}{r}(-k^2+k^2)=0\ ,\quad r\neq 0$$

but 0 is not a δ-function

What about $r=0$?

→ integrate around 0 to get a δ-function

$$\nabla_r G(\mathbf{r},\mathbf{r}')=\frac{-1}{4\pi}\left(\frac{ik}{r}-\frac{1}{r^2}\right)e^{ikr}\,\hat{e}_r$$



↳ allows us to use Gauss' theory

$$\underbrace{\int dV\,\nabla_r^2 G}_{\substack{\text{sphere centered}\\ @\mathbf{r}',\ \text{radius } R}}=\underbrace{\int dS\,\nabla_r G(\mathbf{r},\mathbf{r}')\cdot\hat{n}}_{\substack{\text{same}\\ \text{sphere}}}=\underbrace{4\pi R^2}_{\substack{\text{area of}\\ \text{sphere}}}\left(\frac{-1}{4\pi}\right)\left(\frac{ik}{R}-\frac{1}{R^2}\right)e^{ikR}\xrightarrow[R\to 0]{}1$$

$\Rightarrow \nabla_r^2 G(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')$

the Laplacian everywhere is zero except right where the charge is.

$$\psi_{scat}(\mathbf{r})=\frac{2m}{4\pi\hbar^2}\int d^3\mathbf{r}'\,\underbrace{\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}}\,V(\mathbf{r}')\,\underbrace{e^{ikz'}}$$

z': the z-component of $\mathbf{r}'$

$k\hat{e}_z\cdot\mathbf{r}'=\mathbf{k}_0\cdot\mathbf{r}'$

$(|\mathbf{r}-\mathbf{r}'|)^{-1}\simeq\frac{1}{r}+\dots$

$$k|\mathbf{r}-\mathbf{r}'|=k\sqrt{r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'}$$

$$=kr\sqrt{1-2\frac{\mathbf{r}\cdot\mathbf{r}'}{r}+r'^2/r^2}$$

$$\simeq kr\left[1-\frac{\mathbf{r}\cdot\mathbf{r}'}{r}+\dots\right]=kr-\overbrace{k\mathbf{r}}^{\mathbf{k}}\cdot\mathbf{r}'$$

$$\simeq\frac{-m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int d^3\mathbf{r}'\,V(\mathbf{r}')\,e^{-i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}'}$$

$$\Rightarrow \underbrace{f(\theta,\varphi)}_{\text{direction of }\mathbf{k}}=\frac{-m}{2\pi\hbar^2}\int d^3\mathbf{r}'\,V(\mathbf{r}')\,e^{\underbrace{-i(\mathbf{k}-\mathbf{k}_0)\cdot\mathbf{r}'}_{\text{momentum transfer}}}$$