Lecture 7 - Stark & Zeeman Effects

02/18/16

connection from last lecture:

$\vec{\mu} = \frac{1}{2c}\int \vec{r}\times\vec{j}\,dV$

$\vec{L} = \int \vec{r}\times\vec{p}\,dV$ ← momentum density (mass current)

$\vec{\mu} = \frac{eg}{2mc}\vec{I}$ ← "g-factor"

→ dirac eqn. yields $g_e = 2$

and then, for symmetric matrices. (free particle)

parentheses indicate symmetrization

$\vec{p}\cdot\vec{p} = (\vec{\sigma}\cdot\vec{p})^2 = (\sigma^i p_i)^2 = \underbrace{\sigma^{(i}\sigma^{j)}}_{\equiv\frac{1}{2}(\sigma^i\sigma^j+\sigma^j\sigma^i)} p_i p_j = \delta^{ij} p_i p_j = p^2$

└ this does not hold for all expansions of $\vec{p}$

$(\vec{p} - \frac{e}{c}\vec{A}(x))^2 \neq (\vec{\sigma}\cdot(\vec{p} - \frac{e}{c}\vec{A}))^2$

Conclusion - Hyperfine Structure:

ground state hydrogen → 1s state

the 1s energy level is actually split into a triplet and a singlet

average hyperfine interaction value

$\langle H_{hyp}\rangle = \frac{4}{3} g_N \frac{m}{M_N} (Z\alpha)^4 mc^2 \frac{1}{n^3}\left(\frac{\vec{S}\cdot\vec{I}}{\hbar^2}\right)$

what will change btwn trip & sing.

$(\vec{F})^2 = (\vec{S}+\vec{I})^2 = S^2 + I^2 + 2\vec{S}\cdot\vec{I}$ — $I = \frac{1}{2}$ for proton

$= \frac{3}{2}\hbar^2 + 2\vec{S}\cdot\vec{I} = \vec{F}(F+1)\hbar^2$

$\rightarrow \frac{\vec{S}\cdot\vec{I}}{\hbar^2} = \frac{F(F+1) - 3/2}{2}$

1S:

- $1S_{1/2}, F=1$
- $1S_{1/2}, F=0$

} $\nu = 1420\ \text{MHz}$

↳ $\lambda = 21\ \text{cm}$ (the special 21 cm line for Astronomy gaseous emissions)

Stark Effect (atomic): splitting of levels in an electric field

$\mathcal{H}_1 = e\Phi_{ext} = -e\mathcal{E}z$ ($\vec{\mathcal{E}} = \mathcal{E}\hat{z}$)

($\mathcal{E}$: electric field; z: z-operator)

where the full Hamiltonian for the system is

$\mathcal{H} = \mathcal{H}_{coulomb} + \mathcal{H}_1$

for the ground state $|1s\rangle$ (non-degenerate)

$E^{(1)} = \langle 1s|\mathcal{H}_1|1s\rangle$

$= -e\mathcal{E}\underbrace{\langle 1s|\hat{z}|1s\rangle}$

ground state symmetric in z

$= 0$; no shift in the first ~~second~~ order

- second order perturbation

$$E^{(2)} = \sum_i \frac{|\langle i|\mathcal{H}_1|1s\rangle|^2}{E_0 - E_i}$$

$$= e^2\mathcal{E}^2 \sum_i \frac{|\langle i|\hat{z}|1s\rangle|^2}{E_0 - E_i}$$

$\backsimeq e^2\mathcal{E}^2$

For wavefunctions:

$a \equiv$ Bohr radius

$$\left(\underbrace{\frac{-\hbar^2}{2m}}\nabla^2 - \frac{Ze^2}{r}\right)\psi = E\psi$$

$\left[\frac{\hbar^2}{m}\right] \sim$ energy $\times$ length^2

$e^2 \sim$ energy $\times$ length

$\hookrightarrow \frac{\hbar^2}{me^2} \sim$ length $\equiv a$ (Bohr rad.)

$\backsimeq -a^3\mathcal{E}^2$

actual result: $E^{(2)} = \frac{-9}{4}a^3\mathcal{E}^2$

But what about degenerate perturbation?

Excited States?:

Note: For $\mathcal{E} > 10^3$ V/cm, fine structure smaller than Stark effect
→ neglect fine structure when computing Stark
for $\mathcal{E} \leq 10^3$ V/cm, use J^2 (fine structure) eigenstate as the unperturbed states

excited states ↓

2s, 2p → $|n,\ell,m\rangle = |2,0,0\rangle, |2,1,0\rangle, |2,1,\pm1\rangle$

for 1s

$\langle m'|z|m\rangle = 0$ unless $m' = m$

$\langle m'|[L_z, z] = 0|m\rangle \leftrightarrow (m'-m)\langle m'|z|m\rangle = 0$

for 2S,2p

$\langle 2,1,1|z|2,1,1\rangle = 0$

this is an odd parity state, which acts on $\hat{z}$ to get an even-parity state. This even state has no overlap w/ the odd state to its left

$\langle 2,0,0|z|2,0,0\rangle = 0$

$\langle 2,1,0|z|2,0,0\rangle = -3a$

off-diagonals have non-zero overlap

$\langle 2,\ell',m'|z|2,\ell,m\rangle \leftrightarrow$

$$\left(\begin{array}{cc|cc} 0 & -3a & 0 & \\ -3a & 0 & & \\ \hline 0 & & 0 & 0 \\ & & 0 & 0 \end{array}\right)$$

top left quadrant just $-3a \cdot \sigma_z$ (Pauli matr.)

first sec. eqn

$E^{(1)} = (-e\mathcal{E})(-3a)(\pm 1)$

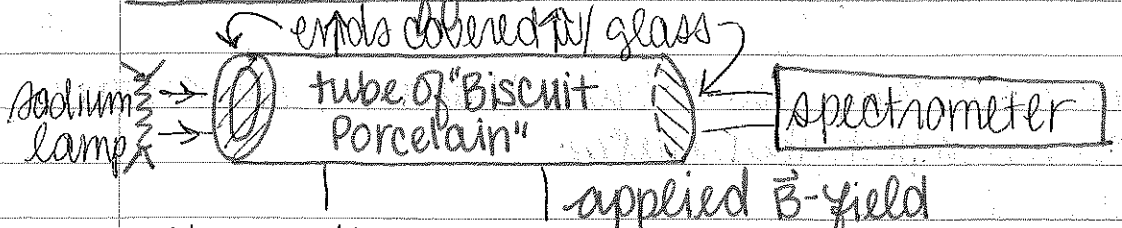
$= \pm 3e\mathcal{E}a \leftrightarrow$ eigenstates → $\frac{1}{\sqrt{2}}(|2,0,0\rangle \pm |2,1,0\rangle)$

these signs match → this is known as the linear Stark effect

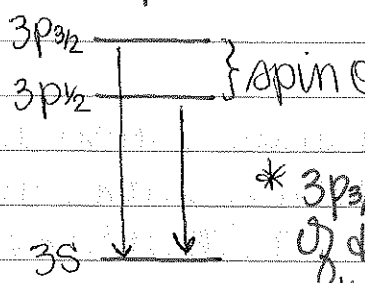
2s & 2p states mix in the presence of an electric field

* You get a dipole moment all time the Stark shift is larger than the split between the levels (0 for degenerate states) → polarize atom as soon as $\mathcal{E}$ is applied)

Zeeman Effect

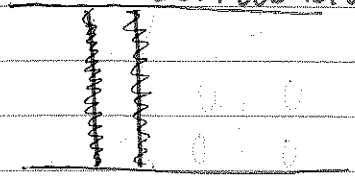


W/ a sodium light source, you have:
the sodium D doublet
$3p \rightarrow 3s$
also experienced fine-structure



* $3p_{3/2} \rightarrow 3s$ has twice the intensity of the $3p_{1/2} \rightarrow 3s$
— there are four states in $p_{3/2}$ between spin & ang. mom, but only two in $p_{1/2}$

Result:
lines broadened when $\vec{B}$ turned on



Zeeman told Lorentz about this, and...
Lorentz predicted: (pre discovery of e⁻!)
① The light @ the edges of the lines is circularly polarized (when $\vec{B}$ parallel to line-of-sight)
② magnitude of broadening determines e/m for the emitting charge (mass to charge ratio)

So how do we calculate this all in QM?
$\mathcal{H}_z = -\vec{\mu} \cdot \vec{B}$ (full mag. mom comes not only from spin but also ang. mom orb.)
$= -\frac{e}{2mc}\left(g_e \vec{S} + \vec{L}\right) \cdot \vec{B}$ ($g_e \approx 2$)

assume $\vec{B}$ in $\hat{z}$ direction

$$= -\frac{e}{2mc}\left(\vec{J} + (g_e - 1)\vec{S}\right)\cdot\vec{B}$$

If $B < 10^6\,G = 10^T$, then Zeeman splitting small compared to fine structure (where fine structure states are eigenstates of J^2)

↳ use $|n\,j\,\ell\,m_j\rangle$ fine-structure eigenstates as unperturbed states

*unperturbed E depends on n, j; not on ℓ or m_j

- need to diagonalize -

$\langle n\,j\,\ell'\,m_j'|H_z|n\,j\,\ell\,m_j\rangle$ ← diagonal here

↳ made up of J_z & S_z $= L_z + S_z$ — cannot change orbital part (ℓ) b/c they do not interact

— J_z already diagonalized

— S_z needs diagonalizing