Lecture 2 - Kinds of Symmetry

02/02/16

1. Rotation Symmetry
2. Translation Symmetry
3. Discrete Translation Symmetry
4. Time-Reversal Symmetry

*You don't always need to evaluate a commutator in order to know if something is a good quantum number.

1. Rotation Symmetry

ex: the Spin operator

$\vec{S}\cdot\vec{p}$, a "pseudoscalar"

↑ because $\vec{S}$ is invariant/scalar under rotation

$\vec{S}$ - orientation dependent

$\vec{p}$ - orientation independent

recall: $\vec{J} = \vec{L} + \vec{S}$ (orbital, spin)

└ total angular momentum

$[J_x, \vec{S}\cdot\vec{p}] = 0$ because $\vec{S}\cdot\vec{p}$ is invariant under rotation (total)

consider J^2: $J^2 = \vec{J}\cdot\vec{J}$

$[J^2, J_z] = 0$ because these are scalars

*If you have a Hamiltonian that commutes with J^2 & J_z, adding something to the Hamiltonian that also commutes does not change the commuting properties of the total system.

where $[J^2, J_z] = 0$

J_z ← $m\hbar$, J^2 ← $j(j+1)\hbar^2$ } eigenvalues

for $|n,j,m\rangle$, $|n,\ell,m\rangle$ eigenvectors

What about L?

$[L_z, \vec{S}\cdot\vec{p}] \neq 0$ because L_z changes $\vec{S}$ but not $\vec{p}$

$[L^2, \vec{S}\cdot\vec{p}] \neq 0$ (L^2 not invariant under translation)

└ where $\vec{p}$ is the generator of translation

ex: "spin-orbit coupling"

$\vec{L}\cdot\vec{S}$, where both $\vec{L}$ & $\vec{S}$ are pseudovectors

$[J^2, \underbrace{\vec{L}\cdot\vec{S}}] = 0 = [J_z, \vec{L}\cdot\vec{S}]$

both don't change sign under rotation together

What about L?

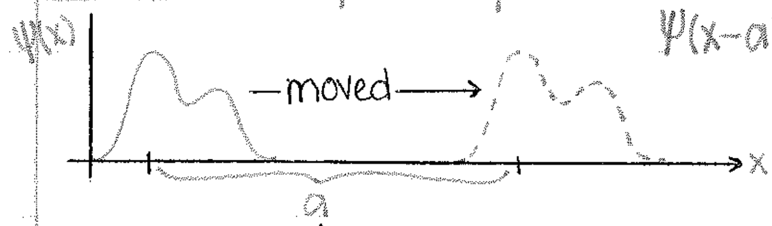
$[L_z, \vec{L}\cdot\vec{S}] \neq 0$ because L_z changes $\vec{L}$ └ not the same defined origin

but unlike the spin operator...

$[L^2, \vec{L}\cdot\vec{S}] = 0$ because $\vec{L}$ generating rotation on the scalar L^2 does nothing

*j, ℓ, j_z all good quantum numbers here

2. Translation Symmetry



distance translated in x

$\psi(x-a) = (U_a \psi)(x)$ └ unitary translation operator

Where $U_a = e^{-ia\hat{p}/\hbar} = e^{-a\underbrace{\frac{d}{dx}}_{ip/\hbar = \frac{d}{dx}}}$

└ $\hat{p}$ is seen here to be the generator of translation

$(U_a\psi)(x) = \left(e^{-a\,d/dx}\psi\right)(x) = \psi(x) - a\psi'(x) + \frac{1}{2}a^2\psi''(x) + \dots$

↓ Recognize Taylor Expansion ↓

$= \psi(x-a)$

*Anything translation invariant will commute with the generator of translation, $\hat{p}$

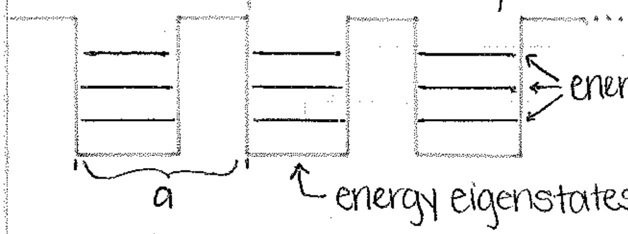
ex: free particle

$\mathcal{H} = \frac{\hat{p}^2}{2m}$ invariant under translation

$[\mathcal{H}, \hat{p}] = 0$

∴ p is a good quantum number

→ we can label the energy eigenstates of the Hamiltonian fully using p ex: Periodic potential



$\mathcal{H} = \frac{p^2}{2m} + V(x)$, $V(x+a) = V(x)$

→ invariant under translation by $n*a$ (multiples of the full translation operator by increments of a)

If $\mathcal{H}|E\rangle = E|E\rangle$

then $\mathcal{H}U_a|E\rangle = U_a\mathcal{H}|E\rangle$

$= EU_a|E\rangle$

↳ they simultaneously commute! ⟺ $[U_a, \mathcal{H}]$

⟹ The state is an eigenstate of both U_a & $\mathcal{H}$. If U is unitary and $\mathcal{H}$ is Hermitian and $[U, \mathcal{H}] = 0$, then U and $\mathcal{H}$ can be simultaneously diagonalized

$U = e^{i\hat{G}}$, where $\hat{G}$ is Hermitian

- If U commutes with $\mathcal{H}$, then G commutes with $\mathcal{H}$ also

3. Discrete Translation Symmetry

Looking at the discrete energy levels of our above periodic potential...

Claim: we can label E-eigenstates by the eigenvalues of U_a since U_a commutes with $\mathcal{H}$

What are the eigenvalues of U_a?

U_a unitary → eigenvalues must be of unit modulus ⟹ pure phases

- eigenvalues $= e^{i\varphi_a}$

But how does φ depend on a?

$\underbrace{(U_a)^n}= U_{na}$

- translating n-times through a is equivalent to translating through na

$\rightarrow (e^{i\varphi_a})^n = \underbrace{e^{in\varphi_a} = e^{i\varphi_{na}}}_{}$

$\varphi_{na} = n\varphi_a + m2\pi$; n & m integers

φ must be linear in a

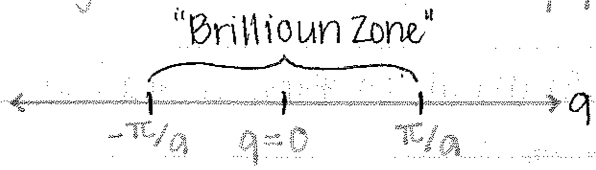
$\Rightarrow$ true for all a $\rightarrow$ $\varphi_a = qa$ for some q

$\therefore$ Eigenvalues of U_a are $\{e^{iqa}\}$; q any real number

note: $(q + 2\pi m/a)$ defines the same eigenvalue of U_a as (q)

$$e^{i(q+2\pi m/a)a} = e^{iqa}\underbrace{e^{i2\pi m}}_{1} = e^{iqa}$$

The eigenvalues of U_a are really quite compact



Any $q > |\pi/a|$ has the same eigenvalues as some q within the Brillioun Zone

$\rightarrow$ q is known as "quasi-momentum" as it is cyclical and does not increase forever like normal momentum

- there may be 1 or >1 E-eigenstates for a given value of q

What does an eigenstate with quasi-momentum q look like?

$\Rightarrow$ Bloch's Theorem

$\psi(x) = e^{iqx} f(x)$

└ the wavefunction can be written as a momentum eigenstate with a specific value of q times some function f(x) (periodic)

$\psi(x-a) = e^{iq(x-a)} f(x-a)$

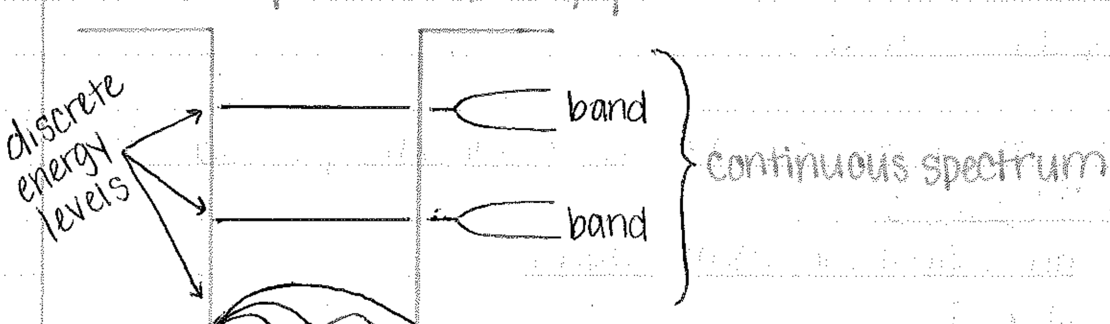
$= e^{-iqa} e^{iqx} f(x-a)$

└ if we want this to be an eigenstate, then

$e^{-iqa} f(x-a) = f(x)$

f(x) must be a periodic function with period a

* If the wells were infinitely deep, and their energies were in levels, these levels will split into bands with their own levels defined by different values of quasimomentum, q.



4. Time-Reversal Symmetry

✓ Time-Dependent Schrödinger Equation

$i\hbar\partial_t\Psi = \mathcal{H}\Psi$

Consider time-reversal

$t \to -t,\ \partial_t \to -\partial_t \Leftrightarrow i \to -i \Leftrightarrow$ complex conjugation

e.g., the momentum eigenstate

$$\left(e^{ipx/\hbar}\right)^* = e^{-ipx/\hbar} = e^{i(-p)x/\hbar}$$

flipping the sign of time should flip the sign of the momentum also

Under time-reversal Θ

- $\vec{p} \to -\vec{p}$
- $\vec{L} = \vec{x} \times \vec{p} \to -\vec{L}$ (sign of $\vec{x}$ doesn't change but sign of $\vec{p}$ does)
- $\vec{S} \to -\vec{S}$

What kind of an operator is Θ?

Θ is not unitary!

– it's not even a linear operator

$(\Theta\Psi)(x) = \Psi^*(x)$

$(\Theta\alpha\Psi)(x) = (\alpha\Psi)^*(x) = \underbrace{\alpha^*}(\Theta\Psi)(x)$

because this scalar is conjugated instead of just popping out unchanged $\Leftrightarrow$ Θ is anti-linear

recall: if unitary

$(Uv, Uw) = (v, w)$

$(\psi_1, \psi_2) = \int \psi_1^* \psi_2\, dx$

$(\psi_1^*, \psi_2^*) = \int \psi_1^{**}\psi_2^*\, dx = (\psi_1, \psi_2)^*$

So, if anti-unitary:

$(\Theta v, \Theta w) = (v, w)^*$

Spin should change sign under time-reversal

spin up $\to$ spin down, etc.

$\to$

How does spin-½ transform under time-reversal?

$\underbrace{\frac{\hbar}{2}\sigma_z}_{S_z}|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$

$S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$

→ Under time reversal:

$|\uparrow\rangle \xrightarrow{\Theta} e^{i\varphi}|\downarrow\rangle$

$|\downarrow\rangle \longrightarrow e^{i\varphi'}|\uparrow\rangle$

* if you rotate a spinor by 360°, it brings back the negative of itself

$(|\uparrow\rangle + |\downarrow\rangle) \rightarrow e^{i\varphi}(|\uparrow\rangle - |\downarrow\rangle)$

└ for S_x, e.g.

⇒ requires that $e^{i\varphi} = -1$!!

⇒ $\Theta^2 = -1$ on the Spin-½ state!
(on a spinor)

$\Theta^2 = -1$ is key!

- for any odd multiple of spin = ½

If $\mathcal{H}$ is invariant under time-reversal & spin is an odd multiple of $\hbar/2$, then $|E\rangle$ & $\Theta|E\rangle$ must be different states

→ $|E\rangle$ & $\Theta|E\rangle$ are degenerate though in the fact that they both have energy eigenvalue E

→ this is known as Kramer's Degeneracy, where for every energy eigenstate of a time-reversal symmetric system with half-integer total spin, there is necessarily at least one more eigenstate with the same energy (every level at least doubly degenerate)

Proof:

If $\Theta|E\rangle = e^{i\delta}|E\rangle$

then $\Theta^2|E\rangle = e^{-i\delta}\Theta|E\rangle = |E\rangle$