Lecture 1 - Introduction & Symmetry

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↳ links to Piazza for discussion

* Read 4.1-2 SAK (Symmetry in Quantum Mechanics)

→ homeworks assigned and due on Thursdays

Textbook: 'Quantum Mechanics', Franz Schwabl

Today

1. Hermitian Operators
2. Symmetry
3. Homework I

1. Warming Up: Hermitian Operators

Observables correspond to $\underbrace{\text{Hermitian}}_{\text{equivalent to self-adjoint for finite-dimensional Hilbert spaces}}$ operators

→ this is equivalent to complex conjugate for numbers

$z^* = z \longleftrightarrow$ real complex number

$\hat{A}^\dagger = \hat{A} \longleftrightarrow$ Hermitian operator, where $\underbrace{\hat{A}}_{\text{Hermitian operator}}\underbrace{|\alpha\rangle}_{\text{"ket" vector in Hilbert space "state"}} = \overset{\text{real \#}}{\alpha}|\alpha\rangle$

↳ eigenvalues are real

↳ $\hat{A} = \sum_\alpha \overset{\text{spectrum of the op.}}{\alpha} \underbrace{|\alpha\rangle\langle\alpha|}_{\text{projection operator, assumed normalized to unity}}$

$$(|\alpha\rangle\langle\alpha|)(|\alpha\rangle\langle\alpha|) = |\alpha\rangle\underbrace{\langle\alpha|\alpha\rangle}_{=1 \text{ if normalized}}\langle\alpha| = |\alpha\rangle\langle\alpha|$$

If $\vec{v}, \vec{w} \in$ Hilbert space:

- $(v,w) \in \mathbb{C}$ - live in complex space
- $(zv,w) = z^*(v,w)$ - anti-linear in this space
- $(v,zw) = z(v,w)$ - linear in this space
- $(v,w)^* = (w,v)$
- $(v,v) \geq 0$ only if $v=0$

In Dirac notation...

$w \longleftrightarrow |w\rangle$ "Ket"

$(v, \cdot) \longleftrightarrow \langle v|$ "bra"; only obtains meaning through inner product

$(v,w) \longleftrightarrow \langle v|w\rangle$

How do we know an operator is Hermitian?

$$(v, \hat{A}w) = (\hat{A}^{\dagger}v, w)$$
$$= (w, \hat{A}^{\dagger}v)^{*}$$

↓ dirac notation ↓

$\langle v|\hat{A}|w\rangle = \langle w|\hat{A}^{\dagger}|v\rangle^{*}$ if $\hat{A}$ is Hermitian

If unitary...

$UU^{\dagger}=1$

In the case of multiple observables of interest:

if $[A,B]=0$ for observables A&B, then they have a common eigenbasis.

$\Rightarrow$ i.e. they are simultaneously diagonalizable

2. Symmetry

Consider a particle of mass m in 1 spatial dimension

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + V(x) \quad , \quad V(-x) = V(x)$$ ⌞ even-parity function

In terms of the parity operator: $(\hat{\pi})$

$$\hat{\pi}\underbrace{|x\rangle}_{} = |-x\rangle$$

a state completely localized at the point x (and therefore non-normalizable)

→ eigenstates of $\hat{\pi}$?

$$\pi(|x\rangle \pm |-x\rangle) = \underbrace{\boxed{\pm}}_{\text{eigenvalues of } \pm 1}(|x\rangle \pm |-x\rangle)$$

$\hat{\pi}^2 = 1 \Rightarrow (\text{eigenvalue of } \pi)^2 = 1 \Rightarrow$ eigenvalues $= \pm 1$ (confirmed)

⌞ $\pi\pi^{\dagger} = \pi^2 = 1$; parity is unitary

In the momentum basis:

$\hat{\pi}|p\rangle = |-p\rangle$ — parity reverses momentum also

$\langle x|p\rangle \propto e^{ipx/\hbar}$

In terms of operators:

$\hat{\pi}\hat{x} = -\hat{x}\hat{\pi}$

$\hookrightarrow \begin{cases} \hat{\pi}\hat{x}|x\rangle = x\hat{\pi}|x\rangle = x|-x\rangle \\ \hat{x}\hat{\pi}|x\rangle = \hat{x}|-x\rangle = (-x)|-x\rangle \end{cases}$

for wavefunctions:

$|\psi\rangle;\ \psi(x) = \langle x|\psi\rangle$

$\langle x|\hat{\pi}|\psi\rangle = \psi(-x)$

$\hat{\pi}\hat{x}\hat{\pi}^{-1} = -\hat{x};\ \hat{\pi}\hat{p}\hat{\pi}^{-1} = -\hat{p}$

↑ multiplying an operator by a unitary operator can still transform the operator in some way

$[\pi, p^2] = \underbrace{[\pi, p]}p + p[\pi, p]$

since $\pi p = -p\pi$,

$[\pi, p] = -2p\pi$

$= -2p\pi p + p(-2p\pi)$

$= -2p\pi p + 2p\pi p$

$= 0$

→ a different approach...

$[\pi, p^2] = \pi p^2 - p^2\pi$

if $\pi p^2 \pi^{-1} = p^2$

$\Longleftrightarrow \pi p^2 = p^2 \pi$

$\Longleftrightarrow [\pi, p^2] = 0$

$\pi p^2 \pi^{-1} \to \pi p \underset{\pi^{-1}\pi}{\wedge} p \pi^{-1}$ (can do because unitary)

$= \underbrace{(\pi p \pi^{-1})}_{-p}\underbrace{(\pi p \pi^{-1})}_{-p}$

$= p^2$ ✓

What about the potential term?

$\pi V(x) \pi^{-1} = V(x)$

if $V(-x) = V(x)$

$\longrightarrow [\pi, V(x)] = 0$

$\therefore [\pi, \mathcal{H}] = 0$

$\Longrightarrow$ this means the eigenstates of $\mathcal{H}$ <u>may be</u> chosen to be parity eigenstates e.g. $\mathcal{H} = p^2/2m$

$\mathcal{H}|p\rangle = (p^2/2m)|p\rangle$ is valid

└ but p is not an eigenstate!

$\hat{\pi}|p\rangle = |-p\rangle$

Can choose instead $(|p\rangle \pm |-p\rangle)$ as the energy eigenstates

(works since $(p)^2 = (-p)^2$)

* this trick does not work if the $\mathcal{H}$-spectrum is non-degenerate

Suppose the $\mathcal{H}$-spectrum is non-degenerate...

and that $\mathcal{H}|E\rangle = E|E\rangle$

$\mathcal{H}\hat{\pi}|E\rangle = \hat{\pi}\mathcal{H}|E\rangle = E\hat{\pi}|E\rangle$

have the same eigenstate

$\Rightarrow \hat{\pi}|E\rangle$ must be $= \pm|E\rangle$ (up to a phase factor $e^{i\varphi}$)

└ each eigenstate $|E\rangle$ must have a definite parity and be non-degenerate

How does parity act on certain observables?

for spherical harmonics:

$\hat{\pi} Y_{\ell m}(\theta,\varphi) = (-1)^{\ell} Y_{\ell m}(\theta,\varphi)$

ex:

$Y_{3m} \longleftrightarrow \sum_{i,j} Z_{ij} x^i x^j$

└ where $Z_{ij} = Z_{ji}$ (symmetric) and $\sum_i Z_{ii} = 0$ (traceless)

where $X^i \longleftrightarrow X = x = r\sin(\theta)\cos(\varphi)$

$X^2 = y = r\sin(\theta)\sin(\varphi)$

$X^3 = z = r\cos(\theta)$

so for 3 X^i's, there will be three (-1) factors, hence why $\hat{\pi}$ changes signs for odd ℓ-values

Parity does not change angular momentum

$\vec{L} = \vec{X} \times \vec{p}$

$\hat{\pi}\vec{L}\hat{\pi}^{-1} = \vec{L}$ — vectors unchanged by parity are known as "pseudo-vectors" or "axial vectors"

$\vec{I} = e^{-i\vec{\theta}\cdot\vec{L}/\hbar}$, a unitary transformation

↖ $\vec{L}$ is the generator of physical rotation

But what about the spin operator, $\vec{S}$?

→ first consider the total rotation of the system, $\vec{J}$

$\vec{J} = \vec{L} + \vec{S}$, $\mathcal{U}(R_{\vec{\theta}}) = e^{-i\vec{\theta}\cdot\vec{J}/\hbar}$, also a unitary transformation

* the parity operator must respect the total quantum-mechanical rotation in the same way that it treats physical rotation (angular momentum), since parity is a reflection through some origin and the total rotation must have the same origin as its components

so since $\hat{\pi}\vec{L}\hat{\pi}^{-1} = \vec{L}$

$\hookrightarrow \hat{\pi}\vec{J}\hat{\pi}^{-1}$ must $= \vec{J}$

from this we know that all components of $\vec{J}$ must hold this same relation

$\Rightarrow \hat{\pi}\vec{S}\hat{\pi}^{-1} = \vec{S}$

3. <u>Homework 1</u>

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + V(\vec{r}) + \lambda\left(\delta^3(\vec{r})\vec{S}\cdot\vec{p} + \underbrace{\text{Hermitian conjugate}}_{\text{needed because } \vec{S}\cdot\vec{p} + \vec{p}\cdot\vec{S}}\right)$$

Which operators commute with the Hamiltonian?

- J^2, J_z → commute with J if $\mathcal{H}$ is invariant under total rotation
  - if $[\mathcal{H}, J] = 0$, then $[\mathcal{H}, f(J)] = 0$ also
- L^2, L_z } check if these are "good Quantum Numbers"
- S^2, S_z }

└ good Quantum Numbers commute with $\mathcal{H}$; we can define the eigenstates of the Hamiltonian using these