09/21/15

Lecture 9 – Specialty Operators

1. Commuting | Compatible Observables
2. Homework 3 (P2)
3. Position Operators
4. Translation
5. Generators

1. Commuting | Compatible Operators

Compatible operators are diagonal in the same basis

$\hat{A} = \sum_n A_n |n\rangle\langle n|$, $\hat{B} = \sum_n B_n |n\rangle\langle n|$

$\Updownarrow$

$[\hat{A}, \hat{B}] = 0$ → if they are compatible, they commute

⇒ also simultaneously diagonalizable

Want to generalize the spectral theorem (the conditions under which an operator or a matrix can be diagonalized)

└ unitary operator → normal operator

Normal Operators

Must satisfy: $NN^{\dagger} = N^{\dagger}N$

define: $H \equiv \frac{(N+N^{\dagger})}{2}$ ← real parts of the operator

$K \equiv \frac{(N-N^{\dagger})}{2}$ ← imaginary parts of the operator

$[H,K] = HK - KH = \ldots = (NN^{\dagger} - N^{\dagger}N)(\ ) = 0$

└ H & K commute

⇒ H & K are Hermitian

↳ simultaneously diagonalizable

$H = \sum_n H_n |n\rangle\langle n|$

$K = \sum_n K_n |n\rangle\langle n|$

$N = H + iK = \sum_n \underbrace{(H_n + iK_n)}_{\text{any complex values}} |n\rangle\langle n|$

Normal Matrix:

↳ $\sum_n Z_n |n\rangle\langle n|$

If the operator is also unitary ⟷ $|Z_n| = 1$

└ must preserve norm

member of eigenbasis

$\left(\sum_n z_n|n\rangle\langle n|\right)|m\rangle = z_m|m\rangle = |\psi\rangle$

must normalize to 1

$\langle\psi|\psi\rangle = |z_m|^2 \underbrace{\langle m|m\rangle}$

$1 \cdot 1 = 1$, as required

2. Homework 3, Problem 2

Infinite-dimensional Hilbert spaces

$\underbrace{UU^\dagger = U^\dagger U}_{=\mathbb{1}}$ (proved in problem 1)

Check that this fails for infinite-dimensional spaces

→ infinite orthonormal basis $\{|n\rangle\}_1^\infty$

↓ theorem from functional analysis ↓

"The Hilbert Space must be "Cauchy Complete" (sequence getting smaller & smaller, there is a point in space where it converges)

3. Position Operators (1.6 SAK)

(for infinite-dimensional Hilbert spaces)

$\hat{x}|x'\rangle = x'|x'\rangle$

Problems:

- positions are not finite
- position is not a countable space (doesn't even really fit in the Hilbert space - we approximate)

Ground rules: continuous operators

($a \to$ discrete, $\xi \to$ continuous)

$\hat{a}|a'\rangle = a'|a'\rangle$

if discrete: $\langle a'|a''\rangle = \delta_{a'a''}$

Kronecker delta

Dirac delta

if continuous: $\langle \xi'|\xi''\rangle = \underbrace{\delta(\xi' - \xi'')}$

non-normalizable, but okay because $|\xi'\rangle$ is not measurable (Heisenberg uncertainty)

- discrete - — continuous -

$\sum_{a'} |a'\rangle\langle a'| = \mathbb{1} \longleftrightarrow \int d\xi' |\xi'\rangle\langle\xi'| = \mathbb{1}$

$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \longleftrightarrow |\alpha\rangle = \int d\xi' |\xi'\rangle \underbrace{\langle\xi'|\alpha\rangle}_{\text{wavefunction}}$

for operator $\hat{A} \longleftrightarrow$ diagonalizable basis

$\langle a'|\hat{A}|a''\rangle = a'\delta_{a'a''} \longleftrightarrow \langle\xi'|\hat{\xi}|\xi''\rangle = \xi'\delta(\xi'-\xi'')$

"delta function: a mathematical rain check on taking an integral"

3D Position

$\underbrace{\hat{x}, \hat{y}, \text{and } \hat{z}}_{\text{simultaneously measurable}}$

$[\hat{x}, \hat{y}] = [\hat{y}, \hat{z}] = [\hat{z}, \hat{x}] = 0$

for $|\vec{r}(x', y', z')\rangle$

$\hookrightarrow \hat{x}|\vec{r}\rangle = x'|\vec{r}\rangle$, etc.

* integrals now over 3 dimensions

4. Translation

(Translation symmetry... if $\hat{T}$ commutes with an operator in its Abelian Group)

$\hat{T}(\vec{\Delta x'})|\vec{x'}\rangle = |\vec{x'} + \vec{\Delta x'}\rangle$

all 3 components (x'_x, x'_y, x'_z)

The translation operator has the structure of an Abelian Group

Identity:

$T(\Delta x' = 0) = \mathbb{1}$

Composition:

$T(\Delta x'')T(\Delta x') = T(\Delta x' + \Delta x'')$

$T(\Delta x')|x'\rangle = |x' + \Delta x'\rangle$

$T(\Delta x'')T(\Delta x')|x'\rangle = |x' + \Delta x' + \Delta x''\rangle$

Inverse:

$T(\Delta x')T(-\Delta x') = \mathbb{1}$

Associative:

$(T(\Delta x''')T(\Delta x''))T(\Delta x') = T(\Delta x''')(T(\Delta x'')T(\Delta x'))$

$\longrightarrow$

Abelian:

$T(\Delta x'')T(\Delta x') = T(\Delta x')T(\Delta x'')$

$\Rightarrow$ It follows from these properties that $T(\Delta x')$ is unitary.

└ unitary follows from conservation of norms (not from Abelian Group properties)

$|\alpha\rangle = \int d\vec{x}' |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$

$|\alpha'\rangle = \hat{T}(\vec{\Delta x}')|\alpha\rangle = \int d\vec{x}' \hat{T}(\vec{\Delta x}')|\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$

$$\langle\alpha'|\alpha\rangle = \langle\alpha|\underbrace{\hat{T}^{\dagger}(\vec{\Delta x}')}_{T^{\dagger}(\Delta x) = T(-\Delta x)}\hat{T}(\vec{\Delta x}')|\alpha\rangle$$

$$= \langle\alpha|\underbrace{\hat{T}(-\vec{\Delta x}')\hat{T}(\vec{\Delta x}')}_{= \mathbb{1}\text{, by inverse property}}|\alpha\rangle = \langle\alpha|\alpha\rangle = 1$$

*Symmetric operators are unitary because they must obey conservation laws.

5. Generators

- Useful for continuous groups

$$\Delta x' = N\underbrace{\left(\frac{\Delta x'}{N}\right)}_{\Delta x''}$$

→ expand $\hat{T}(\Delta x'')$ in a series ↓

without the i, this becomes anti-Hermitian

$T(\Delta x'') = 1 - iK\Delta x'' + \ldots$

K ← generator of translation (Hermitian)

$$T(\Delta x'')T^{\dagger}(\Delta x'') = \mathbb{1}$$
$$= (1 - iK\Delta x'' + \ldots)(1 - iK\Delta x'' + \ldots)^{\dagger}$$
$$= \mathbb{1} - iK\Delta x'' + iK\Delta x'' + \ldots$$

0^{th} order term

first-order terms in K

$(K^{\dagger} - K)\Delta x'' = 0$

$\Rightarrow K = K^{\dagger}$; K must be Hermitian (and will become our momentum)