09/18/15

Lecture 8 – Operator Relations

1. Homework 3 (P1)
2. Eigenvalues/Eigenvectors
3. Degeneracy
4. Commuting/Compatible Operators

1. Homework 3: Problem 1

Getting familiar with [unitary] operators & matrices

unitary operator

$U_{\text{(operator)}} \longleftrightarrow R_{\text{unitary}}$

unitary matrix

$UU^{\dagger} = \mathbb{1} \longleftrightarrow \underbrace{U^{\dagger}U}_{\text{object also identity}} = \mathbb{1}$

$TR(AB) = Tr(BA)$

2. Eigenvalues/Eigenvectors

→ A "constructive" proof that every operator has an eigenvalue and an eigenvector.

$\underbrace{(\hat{A} - \mathbb{1}A_n)}_{\leftrightarrow\ \text{Det}(\hat{A} - \mathbb{1}A_n) = 0\text{, the characteristic polynomial of } A_n}|n\rangle = 0$

– where $\hat{A}$ is the matrix corresponding to $\hat{A}$

$\sum_{\beta} \underbrace{(\hat{A} - \mathbb{1}A_n)}_{M}{}_{\alpha\beta}\, n_{\beta} = 0$

$n_{\beta} = \langle \beta | n \rangle$

Since we know A_n:

$$\left\{\begin{array}{l} M_{11}n_1 + M_{12}n_2 + \ldots = 0 \\ M_{21}n_1 + M_{22}n_2 + \ldots = 0 \\ \text{etc.} \end{array}\right\}$$

N-linear equations for N unknowns $n_{\alpha} = 1 \ldots N$

*Solve by Gaussian elimination

Hermitian Operators

By definition: $\hat{A}^{\dagger} = \hat{A}$

↳ A_i's are real

Claim: If any $A_1 \neq A_2$

→ $\langle 1|2\rangle$ are orthonormal

(where $|1\rangle, |2\rangle$ are eigenvectors of A)

Proof: $\langle 1|\hat{A}|2\rangle = A_2\langle 1|2\rangle$

$\hat{A}$ applied on $|2\rangle$

$= \langle 1|\hat{A}^\dagger|2\rangle = A_1\langle 1|2\rangle$

$\Rightarrow \langle 2|1\rangle = 0$

*Continue proof by $\underbrace{\text{induction}}_{\text{start with 1 and build up}}$ on N — dimension of the matrix

- $N=1 \rightarrow$
- Suppose we prove $N=d$ is true

For $N=d+1$ — find one eigenvector $|d+1\rangle$

$(\hat{A}-\hat{A}_{d+1})|d+1\rangle = 0$

$H_{d+1} = (d+1)$-dimensional Hilbert space

$H_d = \{|\psi\rangle\}: \langle d+1|\psi\rangle = 0$

↳ a "subspace" of the $(d+1)$ Hilbert Space

— Show that this is still a Hilbert Space

$|\psi_1\rangle, |\psi_2\rangle \in H_d$

$c_1|\psi_1\rangle + c_2|\psi_2\rangle \in H_d$ — the space is linear

primes indicate some basis of orthogonal subspace — not eigenvectors of d+1

$\Rightarrow \underbrace{(|1'\rangle, |2'\rangle, |3'\rangle \ldots |d'\rangle)}_{} \cup \{|d+1\rangle\} \in H_{d+1}$

$H_d \rightarrow d$-dimensional — union

Hilbert space that lives in H_{d+1}

*Instead of $\cup$, this could also be $\oplus|d+1\rangle$ — a direct sum such that $H_d \oplus |d+1\rangle = H_{d+1}$ (you can take vectors from both sides of the direct sum and add them up and that is still/also a space)

— Show that $\hat{A}$ is still an operator in H_d

$|\psi\rangle \in H_d \rightarrow \hat{A}|\psi\rangle \in H_d$

$\langle d+1|\hat{A}|\psi\rangle = \langle d+1|\hat{A}^\dagger|\psi\rangle$

since $\hat{A}$ is Hermitian

$= \underbrace{(\hat{A}|d+1\rangle)^\dagger}_{(A_{d+1}|d+1\rangle)^\dagger}|\psi\rangle$

$= A_{d+1}\underbrace{\langle d+1|\psi\rangle}_{\text{orthonormal}} = 0$

$\hat{A} \equiv \bar{A}$ on H_d (d-dim)

↳ we proved it for $N=d$

$\Rightarrow \hat{A}|n\rangle = A_n|n\rangle$, $n=1,\dots,d$

$+\,|d+1\rangle$ such that $\{|1\rangle,\dots,|d+1\rangle\}$ forms a complete set $|j=1,\dots,=d+1\rangle$

Gram-Schmidt Orthonormalization.

$\boxed{N=10 \quad \hat{A}}$ $\quad d+1=10,\ d=9$

$\downarrow$

$\boxed{\hat{A}|10\rangle = A_{10}|10\rangle}$

$\downarrow$ $|1'\rangle$ chosen such that it is orthogonal to $|10\rangle$

$\boxed{|1'\rangle \to \langle 10|1'\rangle = 0;\ |2'\rangle \to \langle 10, 1'|2'\rangle = 0}$

$\downarrow$ $|2'\rangle$ chosen orthogonal to both $|1'\rangle$ & $|10\rangle$

$\boxed{A' = \langle i'|\hat{A}|j'\rangle}$ $\quad$ *You can find up to $|9'\rangle$ this way (for this ex.)

compute $\hat{A}'$ in new basis $\to$ now a 9-dimensional [orthogonal] subspace of A

In this way, we can find / build an entire orthonormal subspace basis to $\{|10\rangle\}$ for which $\hat{A}$ is still an operator.

3. Degeneracy.

Recall:

$\hat{A}|1\rangle = A_1|1\rangle$

$\hat{A}|2\rangle = A_2|2\rangle$

if $A_1 \neq A_2$, then $\langle 1|2\rangle = 0$

this logic does NOT work in reverse

If $\hat{A}$ is Hermitian, then there is an orthonormal basis $\{|n\rangle\}$ such that

$$\hat{A} = \sum_{n=1}^{d+1} A_n |n\rangle\langle n|$$

We know these are real, but we cannot say whether or not they are distinct

It's possible that $A_1 = A_2$ or $A_1 = A_i$

$\Rightarrow$ this is known as a degeneracy

if: $A_1 = A_2 = a$

$\hat{A}|1\rangle = a|1\rangle$

$\hat{A}|2\rangle = a|2\rangle$

$\hat{A}(c_1|1\rangle + c_2|2\rangle) = \underbrace{a(c_1|1\rangle + c_2|2\rangle)} \rightarrow$ an eigenvector

linear combinations become eigenvectors (that are not included in the [orthonormal] set)

if $c_1 = c_2 = 1/\sqrt{2}$

$1/\sqrt{2}\,(|1\rangle + |2\rangle) \not\perp |1\rangle, |2\rangle$

Ambiguity: can choose to create other eigenvectors of like combinations (aka projections)

$|\tilde{1}\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}, \quad |\tilde{2}\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$

where $\hat{A}|\tilde{j}\rangle = a|\tilde{j}\rangle$

To resolve this ambiguity, create an eigensubspace:

$H_{A=a} = \{c_1|1\rangle + c_2|2\rangle\}$

└ 2-dimensional subspace

for $N=3$; $A_3 \neq a$

$\hat{A} = \underbrace{a(|1\rangle\langle 1| + |2\rangle\langle 2|)} + A_3|3\rangle\langle 3|$

$\quad\;\; \parallel$ — both live in the subspace $H_{A=a}$

$= \overbrace{a(|\tilde{1}\rangle\langle\tilde{1}| + |\tilde{2}\rangle\langle\tilde{2}|)} + A_3|3\rangle\langle 3|$

$= \sum_n A_n |n\rangle\langle n|$

└ where $n = 1 \& 2$ are handled by summing the subspace

Generalizing Projection

$\hat{\pi}^2 = \hat{\pi}$; $\hat{\pi}_3 = |3\rangle\langle 3|$

$\hat{\pi}_3^2 = |3\rangle\underbrace{\langle 3|3\rangle}_{1}\langle 3|$

$= |3\rangle\langle 3| = \hat{\pi}_3$

any observable

Such that $Tr(\hat{\pi}_3 O) = \langle 3|O|3\rangle$

$\hat{\pi}_a = |1\rangle\langle 1| + |2\rangle\langle 2|$

$= |\tilde{1}\rangle\langle\tilde{1}| + |\tilde{2}\rangle\langle\tilde{2}|$; $\hat{\pi}_a^2 = \hat{\pi}_a$

*if all eigenvalues are 1 and 0, then π_a is a projection operator and vice versa

General trick for eigenvalues:

$$\pi = \sum_n p_n |n\rangle\langle n|$$

$$\pi^2 = \sum_n p_n^2 |n\rangle\underbrace{\langle n|n\rangle}_{1}\langle n|$$

$$= \sum_n p_n^2 |n\rangle\langle n|$$

$$= \pi \iff p_n^2 = p_n \iff p_n = 0, 1$$

For an infinite-dimensional matrix, if you can show that there is an operator $\hat{\pi} = \hat{\pi}^2$, then you immediately know all the eigenvalues of that matrix are 0 or 1

For Spin 1/2: (Stern-Gerlach)

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_x^2 = S_y^2 = S_z^2 = \left(\frac{\hbar}{2}\right)\mathbb{1}$$

$$\Rightarrow p_n^2 = (\hbar/2)^2 \Rightarrow p_n = \pm \hbar/2$$

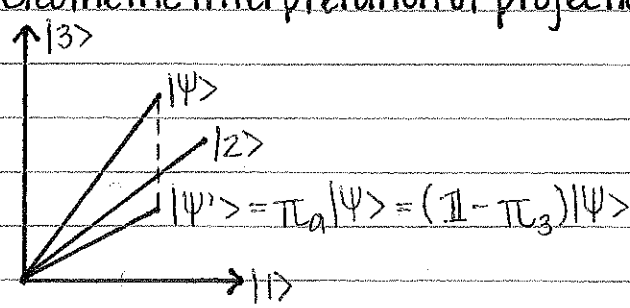
Then for $N=3$; $A_3 \neq a$ (prior example):

$$A = \sum_n A_n |n\rangle\langle n| = \underbrace{a\pi_a + A_3\pi_3}_{\text{no ambiguity!}}$$

Where $\pi_a + \pi_3 = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = \mathbb{1}$

$\rightarrow$ resolution of the identity

Geometric interpretation of projection



$(\mathbb{1} - \pi_3)|\psi\rangle$ is $\perp$ $|3\rangle$

$\pi_3 = |3\rangle\langle 3|$

Generalize:

$|3\rangle \longrightarrow \{|1\rangle, |2\rangle, \ldots, |p\rangle\}$

$\pi_3 \longrightarrow \pi_p = |1\rangle\langle 1| + \ldots + |p\rangle\langle p|$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \llcorner\, p \le N$

$$|\psi\rangle \longrightarrow \underbrace{(\mathbb{1} - \pi_p)|\psi\rangle}_{|\psi'\rangle} \perp (|1\rangle, \ldots, |p\rangle)$$

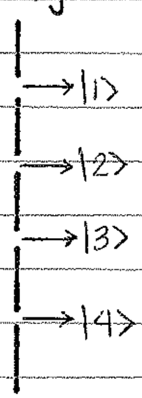
These are orthogonal but they are not normalized

$$|\psi'\rangle \longrightarrow \underbrace{\sqrt{\frac{|\psi'\rangle}{\langle\psi'|\psi'\rangle}}}_{\text{a complete subspace within } \{|1\rangle, \ldots, |p\rangle\}} \Longrightarrow \text{Gram-Schmidt Orthonormalization}$$

Degeneracy / "Incomplete" Measurement

→ splits states into two groups, instead of distinct possible answers

analogous to ~ four-slit interference problem



$$|\psi\rangle = \sum_{j=1}^{4} c_j |j\rangle$$

- turn on detector D1

  $\hat{D}_1 = |1\rangle\langle 1|$

  └ gives eigenvalue 1 if in state $|1\rangle$, 0 otherwise

  $\check{D}_1 = \mathbb{1} - \hat{D}_1$; complementary detector

  └ asks if it went through $|2\rangle, |3\rangle, |4\rangle$

$\hat{D}_1$ and $\check{D}_1$ have degeneracies!

└ $|2,3,4\rangle \Rightarrow 1$

└ $|2,3,4\rangle \Rightarrow 0$

→ **Projection (measurement) postulate:**

(general; for superposition of eigenstates or individual eigenstates)

$$|\psi\rangle \xrightarrow[\check{D}_1 = 1]{\hat{D}_1 = 0} (\pi_{2,3,4})|\psi\rangle$$

$$\pi_{2,3,4} = |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|$$

$A = \sum_\alpha A_\alpha \pi_\alpha$; $\sum_\alpha \pi_\alpha = \mathbb{1}$

If we measure $A \xrightarrow{A_\alpha}$, $|\psi\rangle \rightarrow \pi_\alpha |\psi\rangle$

└ measurement of the observable A in the α-basis projects the waveform into that basis

4. Compatible/Commuting Observables

Two observables are commuting if:

$[\hat{A},\hat{B}]=0$ (they commute, duh!

Compatible: $\hat{A}=\sum_n A_n|n\rangle\langle n|$, $\hat{B}=\sum_n B_n|n\rangle\langle n|$ } ~~diagonal in the same eigenbasis~~

· ~~In the four-slit experiment,~~ $\hat{D}_1$ and $\hat{D}_2$ are compatible

$\hat{D}_1=|1\rangle\langle 1|+0(|2\rangle\langle 2|)+\dots$

$\hat{D}_2=0(|1\rangle\langle 1|)+|2\rangle\langle 2|+\dots$

Show that compatible $\Longleftrightarrow$ commuting (mathematically)

– for compatible $\Rightarrow$ commuting:

$\hat{A}\hat{B}=\dots=\sum_n A_nB_n|n\rangle\langle n|$

$\hat{B}\hat{A}=\dots=\sum_n B_nA_n|n\rangle\langle n|$

$A_nB_n=B_nA_n$

$\Rightarrow \hat{A}\hat{B}=\hat{B}\hat{A}$

– for commuting $\Rightarrow$ compatible:

① Diagonalize $\hat{A}$

$\hat{A}|n\rangle=A_n|n\rangle$

② Look at $\hat{B}$ in $\{|n\rangle\}$

└ n-basis from step ①

$\breve{B}_{nm}=\langle n|\hat{B}|m\rangle$

if $\hat{B}|m\rangle$ is an eigenstate of $\hat{A}$ with the same eigenvalue...

$\hat{A}(\hat{B}|m\rangle)=\hat{B}(\hat{A}|m\rangle)=A_m(\hat{B}|m\rangle)$

└ we can do this because they commute

* it does not matter in which order you observe them

Assume (without loss of generality) that the eigenvalues are ordered

$A_1\le A_2\le\dots\le A_N$

└─ not necessarily distinct, though

$$\breve{B}=\left(\begin{array}{c|c|c} \bar{B}_1 & 0 & \\ \hline & \bar{B}_2 & \\ \hline 0 & & \ddots \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_1\,A_1\,A_1\dots & 0 & \\ \hline & A_2\,A_2\,A_2\dots & \\ \hline 0 & & \ddots \end{array}\right)$$

└ $\breve{B}$ is a block-diagonal matrix

if $A_n\neq A_m$, then $\langle n|(\hat{B}|m\rangle)=0$

i.e., if $\hat{A}$ is not degenerate, then we're done here
($\breve{B}$ already diagonal)

In the diagonal basis: (N=4 example)

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

if $\langle 1|\hat{B}|2\rangle = \langle 2|\hat{B}|3\rangle = \ldots = 0$ (non-degenerate)

$$\breve{B} = \begin{pmatrix} \langle 1|\hat{B}|1\rangle & 0 & 0 & 0 \\ 0 & \langle 2|\hat{B}|2\rangle & 0 & 0 \\ 0 & 0 & \langle 3|\hat{B}|3\rangle & 0 \\ 0 & 0 & 0 & \langle 4|\hat{B}|4\rangle \end{pmatrix}$$

If A has degeneracies... sub-blocks

$$A = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & 0 \\ \hline & & 2 & 0 \\ & 0 & 0 & 3 \end{array}\right) \Rightarrow \breve{B} = \left(\begin{array}{cc|cc} \breve{B}_{11} & \breve{B}_{12} & & \\ \breve{B}_{21} & \breve{B}_{22} & & 0 \\ \hline & & \langle 3|\hat{B}|3\rangle & 0 \\ & 0 & 0 & \langle 4|\hat{B}|4\rangle \end{array}\right)$$

Now $\rightarrow$ diagonalize the sub-blocks

H_1 = subspace $\equiv |1\rangle\langle 1| + |2\rangle\langle 2|$

$= \{c_1|1\rangle + c_2|2\rangle\}$

$$B_{H_1} \rightarrow \begin{pmatrix} \breve{B}_{11} & \breve{B}_{12} \\ \breve{B}_{21} & \breve{B}_{22} \end{pmatrix} \xrightarrow{\text{diagonalize}} |1'\rangle, |2'\rangle$$

We now have $|1'\rangle, |2'\rangle, |3\rangle, |4\rangle$

$$\breve{A} = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & 0 & 2 & \\ & & & 3 \end{pmatrix}, \quad \breve{B} = \begin{pmatrix} \breve{B}_{1'1'} & & & \\ & \breve{B}_{2'2'} & & 0 \\ & 0 & \breve{B}_{33} & \\ & & & \breve{B}_{44} \end{pmatrix}$$