09/16/15

Lecture 7 – Unitary Operators

Brittany's Notes

1. Projection Operator / Measurement / Wavefunction Collapse
2. Unitary Operators
3. Diagonalization

Recap ⇒ Adjoints of operators

Hermitian operators correspond to observables

$A = A^{\dagger}$

→ special subclass: **Positive Operators (P)**

- all expectation values are positive

  $\langle\psi|\hat{P}|\psi\rangle \geq 0$

- P is positive if $\hat{P}$ is Hermitian with all positive p_n

  ↑ eigenvalues of P

  $P|n\rangle = p_n|n\rangle \;,\; p_n \geq 0$

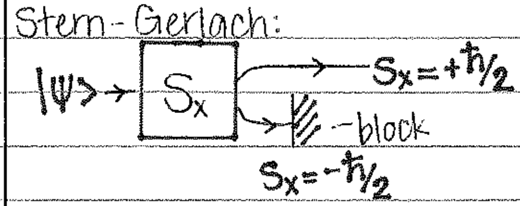
→ sub-subclass: **Projection Operators**

* related to Homework problem 4 *

- these have $p_n = 0$ or 1
- $P^2 = P$
- $P = |\psi\rangle\langle\psi|$

1. Projection Operator / Measurement / Wavefunction Collapse

Stern-Gerlach:



This corresponds to a linear operator (not unitary)

→ call the operator M

$M|\psi\rangle = \alpha|S_x = +\hbar/2\rangle$

→ $M|S_x = +\hbar/2\rangle = |S_x = +\hbar/2\rangle$

→ $M|S_x = -\hbar/2\rangle = 0$

therefore,

$$M = \sum_{\sigma,\sigma' = \pm\hbar/2} |S_x = \sigma\rangle\langle S_x = \sigma|\hat{M}|S_x = \sigma'\rangle\langle S_x = \sigma'|$$

$$= |S_x = \hbar/2\rangle\langle S_x = \hbar/2|$$

2. Unitary Operators

$UU^\dagger = \mathbb{1}$ (identity matrix)

U preserves $\langle | \rangle$

- $|\psi'\rangle = U|\psi\rangle$
- $|\varphi'\rangle = U|\varphi\rangle$

$\langle\psi'|\varphi'\rangle = [U|\psi\rangle]^\dagger [U|\varphi\rangle]$

$= \langle\psi|\underbrace{U^\dagger U}_{1}|\varphi\rangle$

$= \langle\psi|\varphi\rangle$

*#3 of the homework

$U(t)$ for a 2-level atom:

$U(t)|\psi(0)\rangle = |\psi(t)\rangle$

→ choose basis $|E_0\rangle$ & $|E_1\rangle$

Use time-evolution to find elements

$\langle E_j|U(t)|E_k\rangle$ (elements of U)

$U: |\alpha_n\rangle \rightarrow |\beta_n\rangle$ — these are orthonormal bases

$|\beta_n\rangle = U|\alpha_n\rangle$ so $U = \sum_{n,n'} |\alpha_n\rangle\langle\alpha_n|\underbrace{\hat{U}|\alpha_{n'}\rangle}_{|\beta_{n'}\rangle}\langle\alpha_{n'}|$

then $\sum_{n'} \underbrace{\sum_n |\alpha_n\rangle\langle\alpha_n|\beta_{n'}\rangle}_{\text{expansion of } |\beta_{n'}\rangle}\langle\alpha_{n'}|$ so $U = \sum_{n'} |\beta_{n'}\rangle\langle\alpha_{n'}|$

Resolution of $\mathbb{1} = \sum_n |\alpha_n\rangle\langle\alpha_n| = \mathcal{O}$

↳ $\mathcal{O}|\psi\rangle = \underbrace{\sum_n |\alpha_n\rangle\langle\alpha_n|\psi_n\rangle}_{\text{expansion of } |\psi\rangle} = |\psi\rangle$, so $\mathcal{O} = \mathbb{1}$

$|\beta_n\rangle\langle\alpha_n| = U|\alpha_n\rangle\langle\alpha_n|$

→ $\sum_n |\beta_n\rangle\langle\alpha_n| = U\underbrace{\sum_n |\alpha_n\rangle\langle\alpha_n|}_{=\mathbb{1}}$

3. Diagonalization

(eigenvalues & eigenvectors)

$A|n\rangle = A_n|n\rangle$ where A: operator

$|n\rangle$: eigenvector, A_n: eigenvalue } "eigenpair"

Hermitian operators have a complete set of eigenvalues;
$|n\rangle$ is an orthonormal basis
$\Rightarrow$ Observables are Hermitian (this is why)

Practical proof:

① Every operator has one "eigenpair"
— for finite-dimensional space —
- take operator A, pick an orthonormal basis ($|\alpha\rangle$, $|\beta\rangle$, etc.)
- Write A as a matrix
$\hookrightarrow \check{A}_{\alpha\beta} = \langle\alpha|A|\beta\rangle$

② $A|n\rangle = A_n|n\rangle$

$$\langle\alpha|A\underbrace{|n\rangle}_{\text{expand } |n\rangle \text{ in } \alpha,\beta} = A_n\langle\alpha|n\rangle$$

$$\sum_\beta \langle\alpha|A|\beta\rangle\langle\beta|n\rangle = A_n\langle\alpha|n\rangle$$

*could have expanded $|n\rangle$ in β → yields n_β

this translates to the matrix problem
↓ column vectors ↓

$$(\check{A})_{\alpha\beta}(n_\beta) = A_n(n_\alpha)$$

Rewritten...

$$\underbrace{(\check{A} - A_n\mathbb{1})}_{\text{null space}}n_\beta = 0$$

for this to work, the determinant must be 0

$$\det|\check{A} - A_n\mathbb{1}| = 0$$

ex: $\det\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ *for larger matrices, this makes big, messy sums/products

→ Once this is done, you get a polynomial in A_n that will equal zero
$\Rightarrow$ Solve for A_n (our eigenvalues)

example:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det|S_x - \lambda\mathbb{1}| = \det\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0$$

*for anything larger, something like mathematica can be used

$$= \lambda^2 - (\hbar/2)^2 = 0$$

$\lambda = \pm \hbar/2$

*if you have certain symmetries, you can break down the matrix

⇒ To the proof: Fundamental theorem of algebra tells you that you'll get n roots

$$C_n(A_n - \lambda_n)(A_n - \lambda_{n-1}) \ldots (A_n - \lambda_0) = 0$$

$\det|\hat{A} - \lambda \mathbb{1}| = 0$ has at least 1 solution