Lecture 6 - Operators 09/14/15

1. Operators/Matrices
2. Observables

1. Operators/Matrices

$\hat{G}|\psi\rangle = |\psi'\rangle$ ← C_n = column vector representation of $|\psi\rangle$

$= \hat{G}(\sum_n C_n|n\rangle)$

$= \sum_n C_n \hat{G}|n\rangle$

$= \sum_m C_m'|m\rangle$ (I)

← C_m' = column vector representation of $|\psi'\rangle$

→ Multiply by $\langle m|$

$C_m' = \sum_n \underbrace{\langle m|\hat{G}|n\rangle}_{G_{mn}\text{ (matrix)}} C_n$ (II)

2-Level Systems: Stern-Gerlach

$S_x = \hbar/2 \,(|+x\rangle\langle +x| - |-x\rangle\langle -x|)$

↳ write in the z-basis →

$\langle \pm; z|S_x|\pm; z\rangle = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$

$S_x|+y\rangle \longrightarrow C \equiv \langle \pm; z|+; y\rangle$

$= \dots = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$C' \equiv X\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

Matrix ⟶ Vector connotation

kets ⟷ column vector

bras ⟷ row vectors

operators ⟷ matrices

Matrix → Operator

- plug (II) into (I)

$|\psi'\rangle = \sum_m |m\rangle \sum_n \langle m|\hat{G}|n\rangle \underbrace{C_n}_{C_n \equiv \langle n|\psi\rangle}$

$= \sum_{m,n} |m\rangle\langle m|\hat{Q}|n\rangle\langle n|\Psi\rangle$

$= \sum_{m,n} \underbrace{\langle m|\hat{Q}|n\rangle}_{\text{matrix of numbers}} \underbrace{|m\rangle\langle n|\Psi\rangle}_{\text{operator m-n acting on } |\Psi\rangle}$

$\hat{Q} = \sum_{m,n} \langle m|\hat{Q}|n\rangle |m\rangle\langle n|$

Trace

(Analogous to the trace of a matrix)

$\mathrm{Tr}(\underbrace{|\Psi\rangle\langle\varphi|}_{\text{Any operator made out of a bra and a ket}}) \equiv \langle\varphi|\Psi\rangle$ ← inner product

$\Rightarrow \mathrm{Tr}(\hat{Q}) = \sum_{n} \langle n|\hat{Q}|n\rangle$

- Trace is linear

  $\mathrm{Tr}(\hat{A}+\hat{B}) = \mathrm{Tr}(\hat{A}) + \mathrm{Tr}(\hat{B})$

- Matrices/operators inside the trace commute

  $\mathrm{Tr}(\hat{A}\hat{B}) = \mathrm{Tr}(\hat{B}\hat{A})$

2. Observables (A)

For an orthonormal set of states $\{|n\rangle\}$ with some observable A:

Measure A on $|n\rangle \rightarrow$ get A_n

└ associated measurement values

There is an operator, $\hat{A}$, corresponding to making the measurement

$\underbrace{\hat{A}}_{\text{Observables are Hermitian!}}|n\rangle = A_n|n\rangle$

Matrix representation if $\{|n\rangle\}$ is finite:

$A \longleftrightarrow \langle n|\hat{A}|m\rangle$

$= A_m \langle n|m\rangle = \underbrace{A_n\delta_{nm}}_{\text{diagonalized}}$

$$\langle n|\hat{A}|m\rangle = \begin{pmatrix} A_{11} & & 0 \\ & A_{22} & \\ 0 & & \ddots \ A_{nm} \end{pmatrix}$$

$\hookrightarrow \hat{A} = \sum_{n} A_n |n\rangle\langle n|$

Expectation Value $\langle A \rangle$

Measure A on state $|\psi\rangle$

$P(A_n) = |\langle n|\psi\rangle|^2$

The probability of measuring value A_n

$\langle A \rangle = \sum_n A_n P(A_n) = \sum_n A_n \underbrace{\langle\psi|n\rangle\langle n|\psi\rangle}_{|\langle n|\psi\rangle|^2}$

$= \langle\psi|\hat{A}|\psi\rangle$

→ The expectation value is the mean of operator $\hat{A}$

Variance $= \langle A^2\rangle - \langle \hat{A}\rangle^2$

If A is an observable, then any function of A is also an observable

f(A) defined such that:

$f(A)|n\rangle = f(A_n)|n\rangle$

$\langle f(A)\rangle = \langle\psi|f(A)|\psi\rangle$

a function of the observable

Trick:

→ matrix representation

$\langle A \rangle = \hat{\psi}^{\dagger}\hat{A}\hat{\psi}$

row vector $|\psi\rangle$, etc.

ρ = density matrix

$\equiv |\psi\rangle\langle\psi|$

$\langle A \rangle = \mathrm{Tr}(\rho\hat{A})$

$= \mathrm{Tr}(|\psi\rangle\langle\psi|\hat{A})$

$= \mathrm{Tr}(\hat{A}|\psi\rangle\langle\psi|)$

$= \langle\psi|\hat{A}|\psi\rangle$ by definition of Trace

Adjoint

Another way of doing a Hermitian conjugation or conjugate transpose for matrices.

Adjoint → $\dagger$

$A \rightarrow A^{\dagger}$ – adjoint of an operator

$\langle\psi|\hat{A}^\dagger|\varphi\rangle \equiv \langle\varphi|\hat{A}|\psi\rangle^*$

Can check:

$A^\dagger = \sum_{m,n} \underbrace{\langle n|A^\dagger|m\rangle}_{\langle m|A|n\rangle^*} |m\rangle\langle n|$

$\rightarrow (c\hat{A} + c'\hat{B})^\dagger = c^*\hat{A}^\dagger + c'^*\hat{B}^\dagger$

$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$

How to write the adjoint for anything:

Adjoint – general properties

- $(\text{this} + \text{that})^\dagger = \text{this}^\dagger + \text{that}^\dagger$
- $(c \cdot \text{this})^\dagger = c^* \text{this}^\dagger$
- $(\text{this} \cdot \text{that})^\dagger = \text{that}^\dagger \cdot \text{this}^\dagger$
- $(\text{this})^{\dagger\dagger} = \text{this}$

Adjoints for bras & kets

- $|\psi\rangle^\dagger \rightarrow \langle\psi|$
- $\langle\psi|^\dagger \rightarrow |\psi\rangle$
- $\hat{A}^\dagger = [\sum_{n,m} \langle n|\hat{A}|m\rangle(|n\rangle\langle m|)]^\dagger$

$= \sum_{n,m} \langle n|\hat{A}|m\rangle^* \underbrace{(|n\rangle\langle m|)^\dagger}$

$= (\langle m|)^\dagger(|n\rangle)^\dagger$

$= |m\rangle\langle n|$

$= \sum_{n,m} \langle n|\hat{A}|m\rangle^*(|m\rangle\langle n|)$

What is the importance of the adjoint?

Operators can be classified based on adjoint properties.

- if $A = A^\dagger$, A is Hermitian $\longleftrightarrow$ observables
- if $A = -A^\dagger$, A is anti-Hermitian
- if $AA^\dagger = \mathbb{1}$, these are unitary operators
- if $AA^\dagger = A^\dagger A$, these are normal operators

how to write an observable

$A = \sum_n A_n |n\rangle\langle n|$

$A_n \in \mathbb{R}$ (real space) $\longleftrightarrow A_n^* = A_n$

$A^\dagger = \sum_n A_n^* |n\rangle\langle n| = \sum_n A_n |n\rangle\langle n|$, $A^\dagger = A$

Observables $\Longleftrightarrow$ Hermitian Operators