Lecture 5 – Spectra & The Hilbert Space 09/11/15

1. HW1, Problem 3
2. Atomic Spectra
3. The Hilbert Space
4. Operators/Observables
   → Energy (only natural observable)
5. HW2, Problems 1 & 2

1. Math Fact – HW1 #3

$$U_2 = \begin{pmatrix} e^{i\alpha}\cos(\theta) & -e^{i\beta}\sin(\theta) \\ e^{i\gamma}\sin(\theta) & -e^{i\delta}\cos(\theta) \end{pmatrix}$$

└ general 2×2 unitary matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{\dagger} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2+|c|^2 & a^*b+c^*d \\ b^*a+cd^* & |b|^2+|d|^2 \end{pmatrix}$$

$\rightarrow |a|^2+|c|^2 = |b|^2+|d|^2 = 1$

$|a| = \cos(\theta_1)$ $\quad |c| = \sin(\theta_1)$

$|b| = \sin(\theta_2)$ $\quad |d| = \cos(\theta_2)$

$\longrightarrow a^*b = -c^*d \Rightarrow \dfrac{|a|}{|c|} = \dfrac{|d|}{|b|}$

$\Rightarrow \tan(\theta_1) = \tan(\theta_2)$

$\rightsquigarrow \theta_1 = \theta_2$

"Quantum Mechanics is just linear algebra for physicists!"

*Remember: All unitary column vectors have to be orthonormal

$$U(n) = (e_1\ e_2\ \dots\ e_n) = \begin{pmatrix} e_1'^{\dagger} \\ e_2'^{\dagger} \\ \vdots \end{pmatrix}$$

2. Atomic Spectra

Bohr Model (electrons/photons) – the spectra of atoms are very discrete; reason unknown

E_3 ← discrete electron energy levels

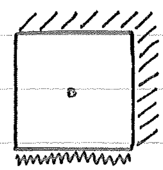
↓ E_2 ⇝ photon $E = (E_3 - E_2)$ released when e^- moves to lower state

E_1

E_0 (moves to a higher state for absorption)

Traditional model → classical jump in electron levels
(electron is always in the levels; when a photon is emitted, the electron just jumped)

Two-Level Atoms
$|E_0\rangle, |E_1\rangle$ -- two kets associated with energy states
• Energy is an "observable"
Isolate an atom -- prevent spontaneous emission



← isolate it by putting it in a box

Because it's an observable:
$|E_{0,1}\rangle$ are exclusive $\Rightarrow \langle E_i|E_j\rangle = \delta_{ij}$ ← i & j are orthonormal
dirac delta
" if in state 0, definitely not in state 1
Newtonian analysis: Spontaneous emission is like friction -- not inherent/a permanent part of the system, but it is not readily obvious that it is removable/preventable.

energy eigenstates
Superposition: $|\psi\rangle = \alpha_0|E_0\rangle + \alpha_1|E_1\rangle$
Probability that energy is in the particular state
$P_{E=E_{0,1}} = |\langle E_{0,1}|\psi\rangle|^2$
general feature of observables
$P_{\theta=u} = |\langle \theta=u|\psi\rangle|^2$
What is the absolute meaning of the energy?
→ energy & frequency are related
Planck: $E = \hbar\omega$ (for photons -- very physical thing)
- for consistency with optics →
$|E_j\rangle \xrightarrow[t]{time} \underbrace{e^{-iE_jt/\hbar}|E_j\rangle}$; $j = 0,1$ -- define time $\omega_j = E_j/\hbar$
the energy eigenstates evolve in frequency under time

Superposition principle $\Rightarrow$ time evolution is linear

$|\psi(0)\rangle = \alpha_0|E_0\rangle + \alpha_1|E_1\rangle$

$\hookrightarrow |\psi(t)\rangle = \alpha_0 e^{-i\omega_0 t}|E_0\rangle + \alpha_1 e^{-i\omega_1 t}|E_1\rangle$

complex constants (assumed nonzero)

- The way an atom radiates is by having an electric dipole that fluctuates $\rightarrow$ dipole "observable"

$|u\rangle = 1/\sqrt{2}\,(|E_0\rangle + |E_1\rangle)$

*Once normalized - remain normalized

$P_u = |\langle u|\psi(t)\rangle|^2$

probability that there is a dipole

$= |\alpha_0 e^{-i\omega_0 t} + \alpha_1 e^{-i\omega_1 t}|^2 \cdot \frac{1}{2}$

$= \frac{1}{2}(1 + 2\mathrm{Re}\{\alpha_0^* \alpha_1 e^{i(\omega_1 - \omega_0)t}\})$

$\Rightarrow$ Oscillating dipole: $\omega = (\omega_1 - \omega_0)$

$\Rightarrow$ photon frequency: $= \omega_1 - \omega_0 = \underbrace{\frac{E_1 - E_0}{\hbar}}_{\text{consistent with Planck}}$

$|\psi(0)\rangle = |E_0\rangle$ -- start in the ground state

$\rightarrow |\psi(t)\rangle = e^{-i\omega_0 t}|E_0\rangle$

$\rightarrow P_u(t) = |\langle u|\psi(t)\rangle|^2 = \underbrace{|\langle u|E_0\rangle|^2}_{\text{constant}}$

Stationary $\rightarrow$ $|E_j\rangle$ are stationary states

they're stuck there when in the box!

$|\psi(t)\rangle = \alpha_0 e^{-i\omega_0 t}|E_0\rangle + \alpha_1 e^{-i\omega_1 t}|E_1\rangle$

analogous to trajectory

$\Rightarrow$ Analog of Newton's law:

$\partial_t|\psi(t)\rangle = -i/\hbar\,[\alpha_0 e^{-i\omega_0 t}E_0|E_0\rangle + \alpha_1 e^{-i\omega_1 t}E_1|E_1\rangle]$

Define: $\hat{H}|E_j\rangle \equiv E_j|E_j\rangle$, the Hermitian "observable"

H is a linear operator - we can find a completely orthonormal set of eigenstates

$\partial_t|\psi(t)\rangle = -i/\hbar\,[\alpha_0 e^{-i\omega_0 t}\hat{H}|E_0\rangle + \alpha_1 e^{-i\omega_1 t}\hat{H}|E_1\rangle]$

$i\hbar\,\partial_t|\psi(t)\rangle = \alpha_0 e^{-i\omega_0 t}E_0|E_0\rangle + \alpha_1 e^{-i\omega_1 t}E_1|E_1\rangle$

The energy observable corresponds with a linear operator H
-- in the sense of $\hat{H}|E_j\rangle = E_j|E_j\rangle$, H is linear

Hermitian operators guarantee these E's are real

Spin is also an observable
-- How? --
$\hat{H} = -\gamma \vec{S} \cdot \vec{B}$

magnetic field you applied

spin

...turn spin into an energy and then you observe it

3. Hilbert Space (Finally!)
define $\mathcal{H} \equiv$ a collection of states

a complex vector space

$|\psi\rangle, |\varphi\rangle \in \mathcal{H}$ exist within

$\alpha|\psi\rangle + \beta|\varphi\rangle \in \mathcal{H}$

within the Hilbert Space

Inner Product:
$\langle\varphi|\psi\rangle \in \mathbb{C}$

exists within complex vector space

$P(\psi \to \varphi) = |\langle\varphi|\psi\rangle|^2$

Probability that state ψ goes to state φ

Linearity:
$\langle\varphi|(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1\langle\varphi|\psi_1\rangle + c_2\langle\varphi|\psi_2\rangle$

$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$

$\langle\varphi|\varphi\rangle > 0$ for all Hilbert Space
- For all physical space, Norm of $|\varphi\rangle > 0$

Orthonormal bases
$\{|\varphi_n\rangle\}$ ← a set of states, φ_n

$\langle\varphi_n|\varphi_m\rangle = \delta_{nm}$

= 1 if n=m, 0 otherwise (Kronecker delta)

$|\psi\rangle = \sum_m C_m|\varphi_m\rangle$

$\rightarrow \langle \varphi_n|\psi\rangle = \sum_m c_m \underbrace{\langle \varphi_n|\varphi_m\rangle}_{\delta_{nm}}$

$\Rightarrow c_n = \langle \varphi_n|\psi\rangle$ $\qquad |c_n|^2 = P(\varphi_n)$

$|\psi\rangle = \sum_m |\varphi_m\rangle\langle\varphi_m|\psi\rangle$ $\qquad$ └ mod-c = Probability you're in state φ_n

Version 1, measurement postulate: $P=|c_n|^2$ $\qquad P=1$

$|\psi\rangle \rightarrow$ measure if in $|\varphi_n\rangle$ or not $\rightarrow |\varphi_n\rangle \rightarrow$ measure again $\rightarrow |\varphi_n\rangle$

└ If you find $|\psi\rangle$ to be in state $|\varphi_n\rangle$, the wavefunction collapses into this state and subsequent measurements will also find $|\varphi_n\rangle$

* Measurement causes the wavefunction to collapse into that measured state!

ex: Want to compute general inner product

$\langle\beta|\alpha\rangle = ?$ $\qquad \vec{\beta}^\dagger\alpha = ?$

Bra-ket $\qquad$ vector

$|\alpha\rangle = \sum_n \alpha_n|\varphi_n\rangle$

$|\beta\rangle = \sum_m \beta_m|\varphi_m\rangle$

$$\vec{\alpha} = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}, \quad \vec{\beta} = \begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_n\end{pmatrix}$$

$$\begin{aligned}\langle\beta|\alpha\rangle &= \sum_{nm} \langle\beta|(\alpha_n|\varphi_n\rangle)\\ &= \sum_n \alpha_n \langle\varphi_n|\beta\rangle^*\\ &= \sum_n \alpha_n[\langle\varphi_n|(\textstyle\sum_m \beta_m|\varphi_m\rangle)]^*\\ &= \sum_{nm} \alpha_n\beta_m^* \underbrace{\langle\varphi_n|\varphi_m\rangle}_{\delta_{nm}}\\ &= \sum_{nm} \alpha_n\beta_m^*\delta_{nm}\\ &= \sum_n \beta_n^*\alpha_n = \beta^\dagger\alpha \leftarrow \text{in the vector space}\end{aligned}$$

Dual Vectors (or bras)

$|\varphi\rangle$ - kets (a vector)

Inner product - braket

$\langle\psi|\varphi\rangle$

$\langle\psi|$ - bras (a dual vector)

ket → bra: $|\psi\rangle = \sum_n c_n|\varphi_n\rangle \rightarrow \langle\psi| = \sum_n c_n^*\langle\varphi_n|$

- The dual vector is the complex conjugate or Hermitian conjugate

$\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$

$= (\sum_n C_n \langle\varphi|\varphi_n\rangle)^*$

$= \sum_n C_n^* \langle\varphi_n|\varphi\rangle$ ✓

4. Operators

for some state $|\psi\rangle$,

$|\psi\rangle \xrightarrow{\hat{G}} \hat{G}|\psi\rangle$

linear operator $\hat{G}$ takes $|\psi\rangle$ into this space

$\hat{G}(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha\hat{G}|\psi\rangle + \beta\hat{G}|\varphi\rangle$

What can we do with operators?

- multiplication

$|\psi\rangle \xrightarrow{\hat{A}} \hat{A}|\psi\rangle \xrightarrow{\hat{B}} \hat{B}(\hat{A}|\psi\rangle)$

$= (\hat{B}\hat{A})|\psi\rangle$ (multiply in opposite order than applied)

- commutation

$[\hat{A},\hat{B}] = (AB - BA)$

Operators are commutable

- linearity

$\underbrace{(\hat{A}+\hat{B})}|\psi\rangle = \hat{A}|\psi\rangle + \hat{B}|\psi\rangle$

If $\hat{A}$ and $\hat{B}$ are operators, then $(\hat{A}+\hat{B})$ is also an operator

Examples of Operators

- Identity operator, $\hat{\mathbb{I}}$

$|\psi\rangle \xrightarrow{\mathbb{I}} |\psi\rangle$

- "Mapping" operator $\hat{G} = |\alpha\rangle\langle\beta|$

$\hat{G}|\psi\rangle = |\alpha\rangle\langle\beta|\psi\rangle = (\langle\beta|\psi\rangle)|\alpha\rangle$

$\hat{G} \rightarrow$ Projection operator when $|\alpha\rangle = |\beta\rangle$

Matrix Representation of Operators

$\{|\varphi_n\rangle\}$ general orthonormal basis

$\hat{G}|\psi\rangle = \sum_n \hat{C}_n |\varphi_n\rangle$

$\check{C}_m = \langle \varphi_m | \hat{G} | \psi \rangle$ must be true due to orthonormality

$= \sum_n \underbrace{\langle \varphi_m | \hat{G} | \varphi_n \rangle}_{\vec{G}_{mn}} C_n$

$\vec{G}_{mn}$ — matrix of the operator

$(\check{C})_m = (\vec{G}_{mn})(C)_n$

Hilbert Space $\longleftrightarrow$ Vector Space

$\|\alpha\rangle$	vector
$\langle\beta\|$	dual vector
$\hat{G}$	matrices

5. HW 2 - Problem 1

$|\alpha_n\rangle \rightarrow$ orthonormal

$\hat{A} = \sum_n \alpha_n |\alpha_n\rangle\langle\alpha_n|$

for $A^2, A^3, \ldots A^n$

$\rightarrow$ Taylor Series (assumes you can)

$f(\hat{A}) = \sum_n f(\alpha_n) |\alpha_n\rangle\langle\alpha_n|$

6. HW2 - Problem 2

Writing the time evolution as an operator for the two-state atom $|E_0\rangle$ & $|E_1\rangle$

$|\psi\rangle = \alpha_0|E_0\rangle + \alpha_1|E_1\rangle \xrightarrow{\text{time op.}} \hat{U}|\psi\rangle$ if linear

$\rightarrow$ Want to write a form for $\hat{U}$ that looks like

$\hat{A} = \sum_n \alpha_n |\alpha_n\rangle\langle\alpha_n|$