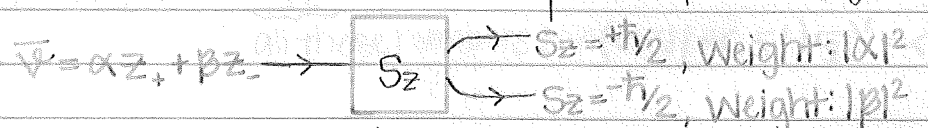
Lecture 4 - Expanding on Stern-Gerlach — 09/09/15

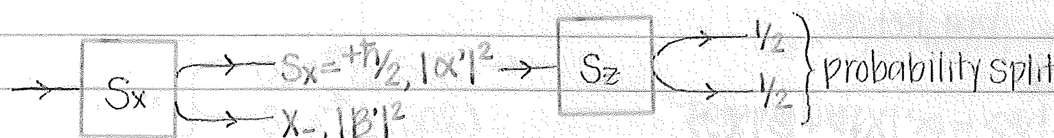
1. Stern-Gerlach 3. Problem #4
2. Bra-ket Notation 4. "Games with Stern-Gerlach" (measurement)

1. Stern-Gerlach

z-component, angular momentum



By doing this measurement, you've determined these particles do have angular momentum



$$X_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \; ; \; X_- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \; ; \; Y_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$$

transformations cannot all be real!

Constraints:

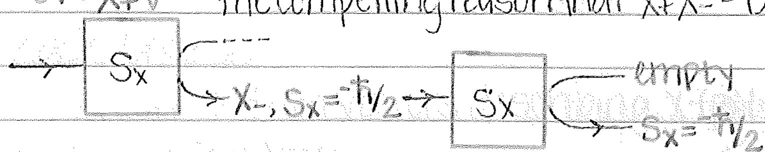
$X_\pm^\dagger X_\pm = 1$; $\underbrace{X_+^\dagger X_- = 0}_{\text{orthogonality}}$

$\Rightarrow \psi = \alpha' X_+ + \beta' X_-$

$|\alpha'|^2, |\beta'|^2$ are the probabilities corresponding to $S_x = \pm\hbar/2$

$X_+^\dagger \psi = \alpha' \underbrace{X_+^\dagger X_+}_{1} + \beta' \underbrace{X_+^\dagger X_-}_{0} = \alpha'$

$\alpha' = X_+^\dagger \psi \rightarrow$ the compelling reason that $X_+^\dagger X_- = 0$



2. Bra-ket Notation

Previously:

$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ → coordinates of position → basis dependent

Want abstract notation (similar to a vector)

- formulated by Dirac

Ket:

$Z_+ \rightarrow \underbrace{|Z_+\rangle}_{\text{ket}}$

for $X_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\Rightarrow |X_+\rangle = \frac{1}{\sqrt{2}}(|Z_+\rangle + |Z_-\rangle)$

Bra:

$\underbrace{\langle\psi| X_+\rangle}_{\text{bra-ket}} = \psi^\dagger X_+$ ← bra

ex:

$|\psi\rangle = \alpha'|X_+\rangle + \beta'|X_-\rangle$

$\alpha' = X_+^\dagger \psi = \langle X_+|\psi\rangle$

$\beta' = X_-^\dagger \psi = \langle X_-|\psi\rangle$ — $|\psi\rangle$ some abstract thing

$\Rightarrow |\psi\rangle = |X_+\rangle\langle X_+|\psi\rangle + |X_-\rangle\langle X_-|\psi\rangle$

└ Pure bra-ket notation: no matrices; basis independent

Starting from a bra-ket, you can go into a coordinate basis easily

→ Pick the $|X_\pm\rangle$ basis

$\vec{\psi} = \begin{pmatrix}\langle X_+|\psi\rangle\\ \langle X_-|\psi\rangle\end{pmatrix}$ — when you enter this column vector notation, your bra-kets are no longer basis independent (you pick the basis → you create the mapping)

3. Problem #4

- to be explained (not key)

$\underbrace{(\hat{S}_x)}_{\text{operator}}|X_\pm\rangle = \pm\hbar/2\,|X_\pm\rangle$

operator - represents measurement by Stern-Gerlach

→ Similarly for $\hat{S}_y|Y_\pm\rangle$ and $\hat{S}_z|Z_\pm\rangle$

Importantly

$X_+^\dagger X_- = \langle X_+|X_-\rangle = 0$ ← orthogonal

→ similarly for y and z

→

$$|\pm;z\rangle = \sum_{j=\pm} U_{ij}(z)|j;x\rangle \quad , \quad |\pm;y\rangle = \sum_{j=\pm} U_{ij}(y)|j;x\rangle$$

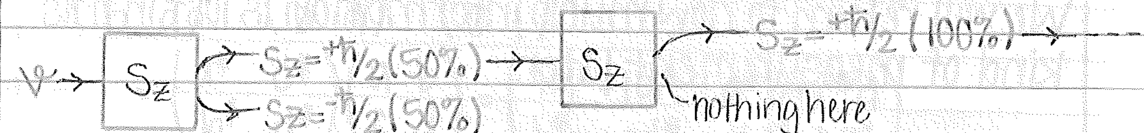
experimental fact: $|U_{ij}(\alpha)| = 1/\sqrt{2}$

$\langle +;z|-;z\rangle = 0$, etc.

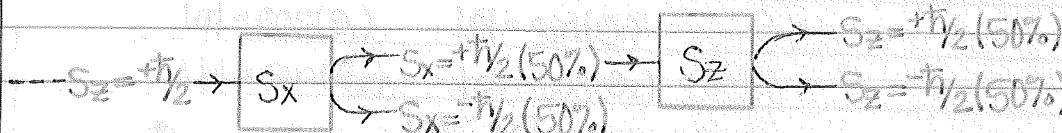
$|\langle \pm;z|\pm;y\rangle = \underbrace{1/\sqrt{2}}$
all these numbers can't be real!

4. "Games with Stern-Gerlach"

Measurement & "Complementarity"



Asking which beam $\Longleftrightarrow$ Asking what is S_z



Measuring S_x made me lose information on S_z!

* S_x and S_z are "complementary variables"
└ idea introduced by Bohr

* Filtering some particles along S_x increases S_z
(No S_{z-} component coming out of 2nd S-G machine → measured states at S_x → now particles in S_{z-} again)

Basis Change

How to switch from vectors describing X-states to vectors describing Z-states.

X-basis: eigenstates of S_x

$$|\psi\rangle = |X_+\rangle \underbrace{\langle X_+|\psi\rangle}_{\psi(X_+)} + |X_-\rangle \underbrace{\langle X_-|\psi\rangle}_{\psi(X_-)}$$

$$\psi(x) = \begin{pmatrix} \langle X_+|\psi\rangle \\ \langle X_-|\psi\rangle \end{pmatrix}$$

Then, just plug in a new basis: $\psi(z)$

$$\psi(z) = \begin{pmatrix} \langle z_+|\psi\rangle \\ \langle z_-|\psi\rangle \end{pmatrix}$$

$$\Rightarrow \langle z_+|\psi\rangle = \langle z_+|x_+\rangle\langle x_+|\psi\rangle + \langle z_+|x_-\rangle\langle x_-|\psi\rangle$$

$$\langle z_-|\psi\rangle = \langle z_-|x_+\rangle\langle x_+|\psi\rangle + \langle z_-|x_-\rangle\langle x_-|\psi\rangle$$

$$\psi(z) = \begin{pmatrix} \langle z_+|x_+\rangle & \langle z_+|x_-\rangle \\ \langle z_-|x_+\rangle & \langle z_-|x_-\rangle \end{pmatrix} \psi(x)$$

$\rightarrow$ You can use this S_x, S_z complementarity for quantum cryptography

We will return to how much information is lost in this kind of measurement later.