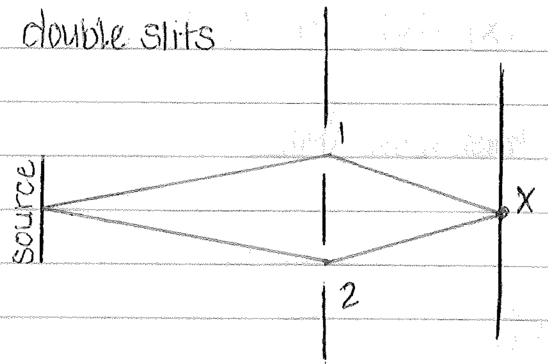
Lecture 3 - Building Up To The Hilbert Space

09/04/15

1. Problem #2
2. Beam Splitters
3. Matrix Methods (normalization/unitary)
4. Problem #3
5. Testing Bombs
6. Spin → Stern-Gerlach
7. Problem #4

1. Problem #2 Interference

double slits



$P(x) = |\varphi_x|^2$

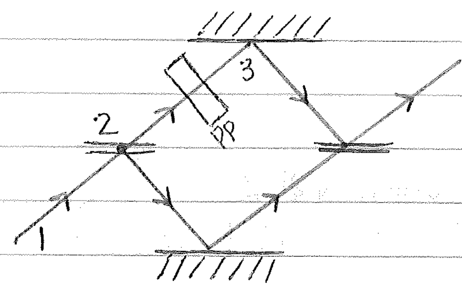
$\varphi_x = \varphi_1 + \varphi_2$

If you measure which slit

→ Classical →

$P(x) = |\varphi_1|^2 + |\varphi_2|^2$

2. Beam Splitters



A wave packet, $v = \binom{\alpha}{\beta}$, as a function of time:

@1 $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ @2 $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ @3 $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$

a definite known state (no superpositions, etc.)

From Lecture 2...

Components of an interferometer:

① Detector $P = |\varphi|^2$

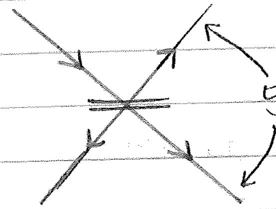
② Phase plate $\begin{pmatrix} e^{i\delta} & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}$

③ Beam Splitter $\begin{pmatrix} w & y \\ x & z \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$

Matrix for a "balanced beam splitter"

→ Splits photons 50/50

→



50% probability of going up or down

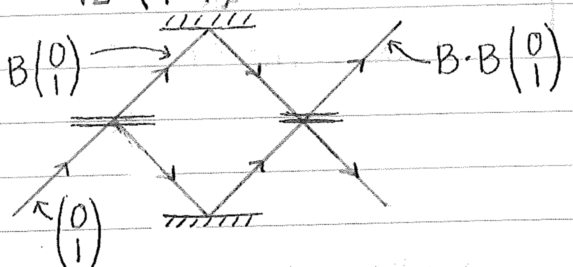
From the top:

$$\begin{pmatrix} W & Y \\ X & Z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} W \\ X \end{pmatrix} \Rightarrow \begin{array}{l} |W|^2 = |X|^2 = \tfrac{1}{2} \quad \xleftarrow{} 50\% \text{ probability} \to \text{mod} = \tfrac{1}{2} \\ |W| = |X| = |Y| = |Z| = 1/\sqrt{2} \end{array}$$

$W = 1/\sqrt{2}$; $X = 1/\sqrt{2}$

- Can $Y = Z = 1/\sqrt{2}$ as well? Can this work? (no)

$$B \stackrel{?}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



$$B \cdot B\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ ← Probability of 2 NOT POSSIBLE (must be < 1)

There must be a sign change somewhere!

$$\Rightarrow B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

* Must constrain norms and norms of sums in Quantum Mechanics problems

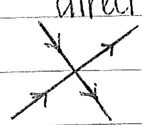
Matrices must "conserve probability"

3. Matrix Methods

(normalization/unitary)

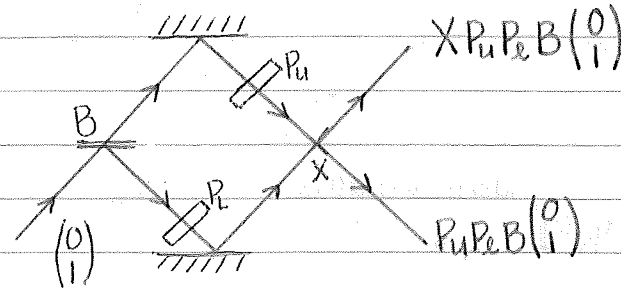
X matrix

└ direction change | phase shift



$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

How do we combine all the interferometer components?



for $\psi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$\psi \to B\psi \to P_uP_\ell B\psi \to XP_uP_\ell B\psi$

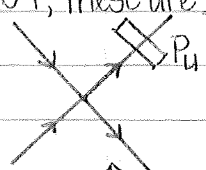
– these are associative!

$X(P_uP_\ell(B\psi)) = \underbrace{(XP_uP_\ell B)}_{\text{represents | describes the interferometer}}\psi$

$P_u = \begin{pmatrix} e^{i\delta_u} & 0 \\ 0 & 1 \end{pmatrix} \qquad P_\ell = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta_\ell} \end{pmatrix}$

$P_uP_\ell = \begin{pmatrix} e^{i\delta_u} & 0 \\ 0 & e^{i\delta_\ell} \end{pmatrix} = P_\ell P_u$ ← the order of your phase plates does not matter

BUT, these are not commutative



$P_uX \neq XP_u$ ⟺ <u>Observables do not commute</u>

cross matrix ⟹ beams intersect; no beam splitter

Normalization

- Can ψ be any matrix? (constraints on 2-state systems)

$\psi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \Rightarrow \underbrace{|\alpha|^2 + |\beta|^2 = 1}_{\text{state vector normalization constraint}}$

$(\alpha^* \ \beta^*)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \psi^\dagger\psi = 1$

†: dagger operator = transpose & complex conjugate

constraint: ψ must normalize to 1

- Can the interferometer be any matrix?

for an interferometer matrix R

$\psi \to R\psi = \psi'$

$\psi'^{\dagger}\psi' = 1$

$\Rightarrow \psi'^{\dagger} = \psi^{\dagger}R^{\dagger}$

$\underbrace{\psi'^{\dagger}\psi'}_{1} = (\psi^{\dagger}R^{\dagger})(R\psi)$

$= \psi^{\dagger}(R^{\dagger}R)\psi$ ← true IFF $R^{\dagger}R = \mathbb{I}$ ← identity matrix

$1 \Rightarrow R^{\dagger}R = \mathbb{I} \Rightarrow$ Unitary R

4. Problem #3

→ prove the converse is true

Show that every R (that is, every unitary matrix) can be made from P_u, P_ℓ, X, B

- Try to get every $\underbrace{\text{real } R}_{\text{orthogonal}}$, then add phase plates

a notation you might choose for the phase plates $= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

5. Testing Bombs

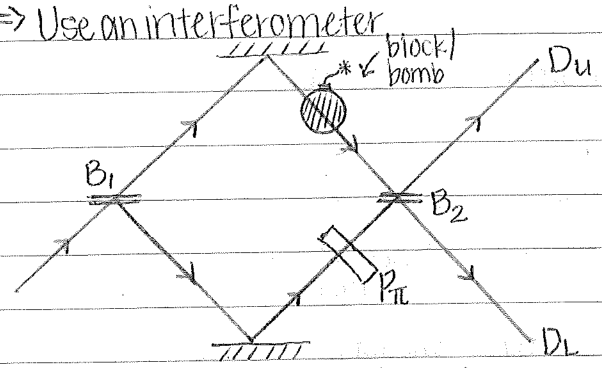
(ex. from 1993; pre-quantum computing)

There are bombs that are photon-activated

→ a single photon sent in... BOOM!!

But, some of the detectors don't work. How do you figure out which detectors work without exploding all the bombs?

⇒ Use an interferometer



$B_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$B_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$B_1 P_\pi B_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{P_\pi}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\underbrace{\begin{pmatrix}1&1\\-1&1\end{pmatrix}}_{P_\pi B_2} = \underset{\uparrow\,\text{unitary}}{\begin{pmatrix}0&1\\-1&0\end{pmatrix}} = R$$

only valid/applicable in the quantum sense (ignore when in classical conditions)

$R\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ – a photon shot in from the bottom branch comes out on top branch (nothing to D_L)

If we block the upper beam (classical → we know all photons in lower leg)
↳ 1/4 of the photons exit on the lower beam
  └ 50% probability with 50% of the photons

Now, we replace the block with a bomb (capacity unknown)
= detector/observer if functional

- Photon → upper leg — You know it went through upper leg now
  → [working] bomb → *BOOM* (w/ 50% chance)
- Photon → lower leg — You know it went through lower leg now
  → [working] bomb doesn't explode → D_u, D_L clicks (D_u = 1/4, D_L = 1/4)

By doing this experiment with the <u>working</u> bomb, interference is destroyed by the "observation" of which path the photon took.

Working Bomb ⇒ forces to classical solution
  └ i.e. R not applicable

If the bomb is <u>not</u> working ⇒ quantum solution
→ click only at D_u (as a result of R... or by chance)

<u>Possibilities</u>

Bomb
- working
  - 1/2 → upper leg → Bomb explodes
  - 1/2 → lower leg (now classical)
    - 1/2 → $D_\ell = 1/4$
    - 1/2 → $D_u = 1/4$
- not working → quantum case
  - $D_\ell = 0$
  - $D_u = 1$

⇒ Click on D_ℓ = definitive working bomb (without exploding it!)

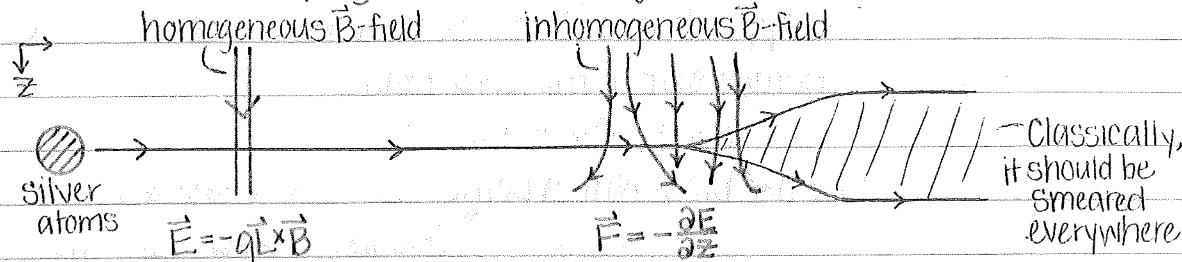
6. Spin: Stern-Gerlach

"basis independence"

· Angular momentum is quantized

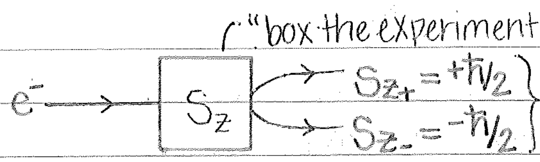
↳ this is a prediction of the Bohr Model of an atom

Experiment: Trying to measure angular momentum



⇨ Despite classical expectations, the angular momentum was quantized into discrete bands

The Stern-Gerlach Machine



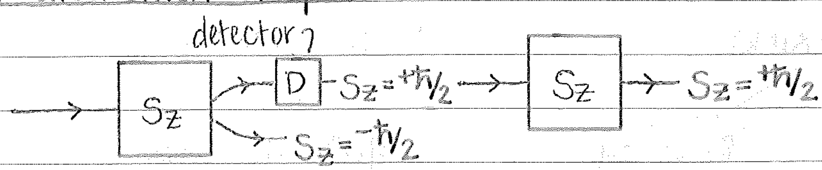
electron splits into two discrete beams

$L \to S$: Angular Momentum becomes Spin

$$\psi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha\underbrace{\left(S_z = \frac{+\hbar}{2}\right)}_{\begin{pmatrix}1\\0\end{pmatrix} = Z_+} + \beta\underbrace{\left(S_z = \frac{-\hbar}{2}\right)}_{\begin{pmatrix}0\\1\end{pmatrix} = Z_-}$$

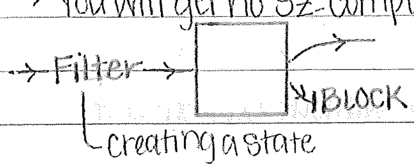
What's special about z? Nothing! The Universe is rotationally invariant.
(i.e. You can make a Stern-Gerlach Machine along X, Y, etc.)

Measurement of Spin



Measurement of $S_z = +\hbar/2$ collapses the waveform into this state

⇒ You will get no S_z-component from a second Stern-Gerlach$_z$



Rotate to x:

$\rightarrow S_z = \frac{+\hbar}{2} \rightarrow \boxed{S_x}$ $\rightarrow S_x = +\hbar/2$, $\rightarrow S_x = -\hbar/2$ } even with an input of solely $S_z = \frac{+\hbar}{2}$!

$Z_+ = \underbrace{\alpha'(S_x = +\hbar/2)}_{X_+} + \underbrace{\beta'(S_x = -\hbar/2)}_{X_-}$

"The notation is just going to get more inconvenient until we give into using bras & kets".

$|\alpha'|^2 = |\beta'|^2 = 1/\sqrt{2}$ (50/50 split)

In reverse...

$\rightarrow S_x = \frac{+\hbar}{2} \rightarrow \boxed{S_z}$ $\rightarrow S_z = +\hbar/2$, $\rightarrow S_z = -\hbar/2$

$X_+ = \alpha(S_z = \hbar/2) + \beta(S_z = -\hbar/2)$; $|\alpha|^2 = |\beta|^2 = 1/\sqrt{2}$

Given a state, you can choose its phase <u>once</u>

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}} Z_+ + \frac{1}{\sqrt{2}} Z_-$

$X_- = \alpha_-(S_z = +\hbar/2) + \beta_-(S_z = -\hbar/2)$

$|\alpha_-|^2 = |\beta_-|^2 = 1/\sqrt{2}$

In analogy with the beam splitter matrix:

to conserve probability, if $\alpha = \beta = 1/\sqrt{2}$

$\Rightarrow \alpha_- = -\beta_- = \frac{e^{i\delta}}{\sqrt{2}} \rightarrow 1$

β_- must be $\propto 1/\sqrt{2}$

$X_\pm = \frac{Z_+ \pm Z_-}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} Z_\pm$

7. Problem #4

→ repeat all this with y

hint: introducing S_y creates complex numbers

$\rightarrow S_x = \frac{+\hbar}{2} \rightarrow \boxed{S_y}$ $\rightarrow S_y = +\hbar/2$ (50%), $\rightarrow S_y = -\hbar/2$ (50%)

$Y_\pm = (\alpha'_\pm Z_+ + \beta'_\pm Z_-)$

$\Rightarrow \alpha'_\pm$ and $\beta'_\pm$ can't be wholly real

$Y_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix} Z_\pm$ this choice is not unique for y, but there is no choice that allows them to be real

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

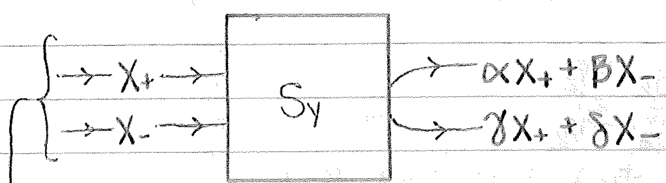
8. <u>Orthogonality</u>

ex: $\chi_+^\dagger \chi_- = 0$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = 0$$ orthogonal!

Prove this using known normalization

Normalization constraint: $\chi_+^\dagger \chi_+ = \chi_-^\dagger \chi_- = 1$



Combination must be normalized

$$(\alpha\chi_+ + \beta\chi_-)^\dagger(\alpha\chi_+ + \beta\chi_-) = |\alpha|^2 \underbrace{\chi_+^\dagger\chi_+}_{1} + |\beta|^2 \underbrace{\chi_-^\dagger\chi_-}_{1} + (\alpha^*\beta\chi_+^\dagger\chi_- + \alpha\beta^*\chi_+\chi_-^\dagger)$$

Where $|\alpha|^2 + |\beta|^2 = 1$

$$= \underbrace{|\alpha|^2 + |\beta|^2}_{1} + (\alpha^*\beta\chi_+^\dagger\chi_- + \alpha\beta^*\chi_+\chi_-^\dagger)$$

$$= 1 + \underbrace{(\alpha^*\beta\chi_+^\dagger\chi_- + \alpha\beta^*\chi_+\chi_-^\dagger)}_{\text{so this must} = 0} = 1$$ constraint

$$\Rightarrow 2\mathrm{Re}\{\underbrace{\alpha^*\beta}_{}\chi_+^\dagger\chi_-\} = 0$$

if I can change phase...

$$\Rightarrow \chi_+^\dagger\chi_- = 0$$