12/02/15

Lecture 39 – Angular Momentum Addition (3.8 SAK)

Addition of $\vec{L}$

(Applying the same rotation to multiple systems)

Canonical application:

Hydrogen atom; for the spin $= \frac{1}{2}$ e^-

Hilbert space $\underbrace{|x'\rangle \otimes |\pm\rangle}_{\text{tensor product space of the position-space and spin-space bases}}$

What are the wavefunctions for this space?

$$(\langle x'| \otimes \langle \pm|)|\psi\rangle = \psi_\pm(x')$$

$$\hookrightarrow |\psi\rangle = \sum_{x', \sigma_z = \pm} \underbrace{\psi_{\sigma_z}(x')}_{\text{wavefunctions}} |x'\rangle \otimes |\sigma_z\rangle$$

Conventionally:

$$\tilde{\psi}(x') = \begin{pmatrix} \psi_+(x') \\ \psi_-(x') \end{pmatrix} \rightarrow$$ Spinor; transforms like the spin-$\frac{1}{2}$ state

transforms as

$$\tilde{\psi}(x') \xrightarrow[\text{rotation}]{\text{spin}} e^{-i/\hbar \varphi \hat{n} \cdot \vec{S}} \tilde{\psi}(x')$$

$\tilde{\psi}$ – the spinor

$\vec{S}$ – spin-½ matrices; does not affect position

What about the total rotation?

(vs. just spin rotation, like above)

$$\mathcal{D}(R)|\psi\rangle = \sum_{x' \sigma_z} \psi_{\sigma_z}(x') \mathcal{D}(R)(|x'\rangle \otimes |\sigma_z\rangle)$$

$$= \sum_{x' \sigma_z} \psi_{\sigma_z}(x') (\mathcal{D}_{orb}(R)|x'\rangle) \otimes (\mathcal{D}_{spin}(R)|\sigma_z\rangle)$$

$\mathcal{D}_{orb}$ – orbital (position) rotation

$\mathcal{D}_{spin}$ – spin rotation

$$\rightarrow \mathcal{D}_{orb}(R) = \exp(-i\varphi/\hbar \, \vec{L} \cdot \hat{n})$$

$$\rightarrow \mathcal{D}_{spin}(R) = \exp(-i\varphi/\hbar \, \vec{S} \cdot \hat{n})$$

$$\Rightarrow \mathcal{D}(R)(|x'\rangle \otimes |\sigma_z\rangle) = [\exp(-i\varphi/\hbar \, \vec{L} \cdot \hat{n})|x'\rangle] \otimes [\exp(-i\varphi/\hbar \, \vec{S} \cdot \hat{n})|\sigma_z\rangle]$$

$\longrightarrow$

for the spin rotation component:

$$|x'\rangle \otimes (\exp(-i\varphi/\hbar\, \vec{S}\cdot\hat{n})|\sigma_z\rangle)$$
$$= (\underbrace{\mathbb{1}}_{\text{Identity tensor}} \otimes \exp(-i\varphi/\hbar\, \vec{S}\cdot\hat{n}))(|x'\rangle \otimes |\sigma_z\rangle)$$

$\Rightarrow \exp(-i\varphi/\hbar\, (\mathbb{1}\otimes\vec{S})\cdot\hat{n})$, **the spin rotation operator!**

$$= \mathbb{1} - i\varphi/\hbar\, (\mathbb{1}\otimes\vec{S})\cdot\hat{n} + \mathbb{1}\otimes(-i\varphi/\hbar\, \vec{S}\cdot\hat{n})$$

similarly for the orbital component:

$$\mathcal{D}(R)(|x'\rangle\otimes|\sigma_z\rangle) = \exp(-i\varphi/\hbar\, \underbrace{(\vec{L}\otimes\mathbb{1})\cdot\hat{n})\exp(-i\varphi/\hbar\, (\mathbb{1}\otimes\vec{S})}_{\text{these commute!}}\cdot\hat{n})|x'\rangle\otimes|\sigma_z\rangle$$

$$= \exp(-i\varphi/\hbar\, \underbrace{(\vec{L}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{S})}\cdot\hat{n})|x'\rangle\otimes|\sigma_z\rangle$$

$=\vec{J}$, the representation of the total rotation in the vector product space in both spin and position

$$\vec{J} = \underbrace{\vec{L}\otimes\mathbb{1}_{\sigma_z}}_{\vec{L}} + \underbrace{\mathbb{1}_{x'}\otimes\vec{S}}_{\vec{S}}$$

$$\vec{L}\to\vec{J}_1 \qquad \vec{S}\to\vec{J}_2$$

$$\vec{J}_{tot} = \underbrace{\vec{J}_1\otimes\mathbb{1}}_{\vec{J}_1} + \underbrace{\mathbb{1}\otimes\vec{J}_2}_{\vec{J}_2}$$ — where $\vec{J}_1$ & $\vec{J}_2$ are fixed for the entire rotation

$[\vec{J}_1, \vec{J}_2] = 0$, **where $\vec{J}_1, \vec{J}_2, \vec{J}_{tot}$ all obey angular momentum commutation relations**

Basis option:

$$\left.\begin{aligned} J_a^2|j_a, m_a\rangle &= j_a(j_a+1)\hbar^2|j_a, m_a\rangle \\ J_{az}|j_a, m_a\rangle &= m_a\hbar|j_a, m_a\rangle \end{aligned}\right\} a = 1, 2, tot$$

→ **the natural choice for this basis is the tensor product basis between 1 & 2**

the product basis $= \underbrace{|j_1, m_1\rangle}_{\equiv |m_1\rangle} \otimes \underbrace{|j_2, m_2\rangle}_{\equiv |m_2\rangle}$

$$|m_1\rangle\otimes|m_2\rangle = |m_1, m_2\rangle$$

***But these are NOT J_{tot}^2 eigenstates!**

→ Rotation properties are complicated and we must have an alternate basis to represent the total rotation

Alternate basis: $|j_{tot}, m_{tot}\rangle$

→

$|j_{tot}, m_{tot}\rangle = \sum_{m_1, m_2} |m_1, m_2\rangle \underbrace{\langle m_1, m_2 | j_{tot}, m_{tot}\rangle}$

Clebsch-Gordan Coefficients -- allow you to move between product basis and total basis

Calculation of C-G coefficients in principle:

$J_{tot}^2 = J_1^2 + J_2^2 + 2J_1 \cdot J_2$ (in the $|m_1, m_2\rangle$ basis)

$\rightarrow$ diagonalize $\rightarrow$

yields $|j_{tot}, m_{tot}\rangle$ as eigenstates

What about $\vec{J}_{tot,z}$?

$\vec{J}_{tot,z} |j_{tot}, m_{tot}\rangle = m_{tot}\hbar |j_{tot}, m_{tot}\rangle$

$= \sum_{m_1 m_2} \underbrace{(\vec{J}_{1,z} + \vec{J}_{2,z})}_{=(m_1\hbar + m_2\hbar)} |m_1, m_2\rangle\langle m_1, m_2 | j_{tot}, m_{tot}\rangle$

$\Rightarrow$ **$m_{tot} = m_1 + m_2$ or the Clebsch-Gordan coefficients vanish**

What does this say about $\vec{J}_{tot,z}$? That you can only move into this basis under certain conditions?

Triangle rule: $j_1 - j_2 \leq j_{tot} \leq j_1 + j_2$

$m_1 \leq j_1$; $m_2 \leq j_2 \Rightarrow m_{tot} = m_1 + m_2 \leq j_1 + j_2$

$\rightarrow j_{tot}^{MAX} = j_1 + j_2$ and similarly $j_{tot}^{MIN} = |j_1 - j_2|$

*Next lecture: Why there is only one j_{tot} for each allowed value of E