11/30/15

Lecture 38 – Spherical Vectors and the 3D Harmonic Oscillator

1. 3D Spherical Harmonic Oscillator
2. Spherical Vectors
3. Applying Spherical Vectors to the 3D SHO

1. 3D Spherical Harmonic Oscillator

Take the obvious set of eigenstates

$|n_x, n_y, n_z\rangle$

$\hookrightarrow N_{tot} = n_x + n_y + n_z$

$\frac{1}{2}(N_{tot}+1)(N_{tot}+2)$ – fold degeneracies

and relate it to the eigenstates in the angular momentum basis

$|N_{tot}, \ell, m\rangle$

last time: p=1

$$|N_{tot}=2, \ell=0=m\rangle = \tfrac{1}{\sqrt{3}}\underbrace{(|n_x=2, n_y=0, n_z=0\rangle + |n_x=0, n_y=2, n_z=0\rangle + |n_x=0, n_y=0, n_z=2\rangle)}_{\text{symmetric superposition – a rotationally invariant state}}$$

What about for the $\ell=1$ case? We will need...

2. Spherical Vectors

For Cartesian vector operators

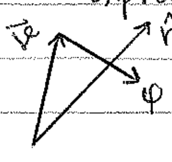
$$\mathcal{D}(R)\, V_\alpha\, \mathcal{D}^\dagger(R) = \sum_\beta R_{\alpha\beta} V_\beta \quad (\text{I})$$

$\rightarrow$ Under infinitesimal rotation $(\hat{n}, \varphi)$, $\hat{n} = (n_x, n_y, n_z)$ ⤵

$$\mathcal{D}(R) = \exp\left(\frac{-i\varphi}{\hbar} \sum_\gamma J_\gamma n_\gamma\right) = e^{-i(\vec{J}\cdot\hat{n})\varphi/\hbar} \simeq \mathbb{1} - i\vec{J}\cdot\hat{n}\frac{\varphi}{\hbar} \quad (\text{II})$$

$$R_{\alpha\beta} \simeq \delta_{\alpha\beta} - \varphi \sum_\gamma \varepsilon_{\alpha\beta\gamma} n_\gamma \quad (\text{III})$$

└ expression of infinitesimal rotation



$$\frac{\delta\vec{r}}{\delta t} = v = \underbrace{\vec{\omega}\times\vec{r}}_{\delta\vec{v} = \varphi\hat{n}\times\vec{v}} = \hat{n}\frac{d\theta}{dt}\times\vec{r}$$

$$\delta\vec{r} = (\hat{n}\times\vec{r})\delta\theta = -(\vec{r}\times\hat{n})\delta\theta$$
$$= R\vec{r} - \vec{r} = (R-\mathbb{1})\vec{r}$$
$$= (R-\mathbb{1})_{\alpha\beta} r_\beta = -\delta\theta\, \varepsilon_{\alpha\beta\gamma} r_\beta n_\gamma$$

which holds for any $r, \hat{n}$ and $\delta\theta$

$\rightarrow$ Applying (I)+(II) to (III) ⤵

$$\cancel{V_\alpha} - \frac{i\cancel{\varphi}}{\hbar}\sum_\gamma n_\gamma [J_\gamma, V_\alpha] = \cancel{V_\alpha} - \cancel{\varphi}\sum_\gamma n_\gamma \sum_\beta \varepsilon_{\alpha\beta\gamma} V_\beta$$

- equate coefficients of n_γ

$[J_\gamma, V_\alpha] = -i\hbar \sum_\beta \varepsilon_{\alpha\beta\gamma} V_\beta$ ← looks like commutator $[J_\alpha, J_\beta] = i\hbar\varepsilon_{\alpha\beta\gamma} J_\gamma$ because the angular momentum operator is also a vector operator

↓ more generally ↓

$[J_\alpha, V_\beta] = i\hbar \sum_\gamma \varepsilon_{\alpha\beta\gamma} V_\gamma$

These are similar steps to angular momentum algebra

$\Rightarrow (V_x, V_y, V_z) \longrightarrow (V_+, V_-, V_z)$

define: $J_\pm = J_x \pm iJ_y$; $V_\pm = V_x \pm iV_y$

$[J_+, V_-] = -[J_-, V_+] = 2\hbar V_z$

$[J_+, V_+] = [J_-, V_-] = 0$

these commutation relations satisfied by (V_-, V_z, V_+) define a vector in the "spherical basis" (or spherical vectors)

- Using angular momentum algebra, we can check:

$$\left\{\begin{array}{l} [J_+, V_-] = -[J_-, V_+] = 2\hbar V_z \\ [J_+, V_+] = [J_-, V_-] = [J_z, V_z] = 0 \\ [J_\pm, V_z] = \mp \hbar V_\pm \\ [J_z, V_\pm] = \pm \hbar V_\pm \end{array}\right\}$$

These form the definition of a spherical vector

→ $V_\pm$ raises/lowers J_z by $\pm\hbar$, etc.

Check that these are commuting raising/lowering operators:

$$J_z V_\pm |l,m\rangle = \underbrace{V_\pm J_z |l,m\rangle}_{m\hbar V_\pm |l,m\rangle} + \underbrace{[J_z, V_\pm]}_{\pm\hbar V_\pm} |l,m\rangle$$

$= (m\pm1)\hbar V_\pm |l,m\rangle$, **$V_\pm$ increases/decreases m by one**

↑ ✓ these act as raising/lowering operators

(note: V_z does <u>not</u> though)

3. Applying Spherical Vectors to the 3D SHO

$\left(a^\dagger_\pm = \dfrac{a^\dagger_x \pm i a^\dagger_y}{\sqrt{2}},\ a^\dagger_z\right)$ are spherical vectors

where $[a^\dagger_\alpha, a_\beta] = \delta_{\alpha\beta}$ for $\alpha = \pm, z$

Recall that

$N_{tot} = (n_x + n_y + n_z) = \sum_\alpha a^\dagger_\alpha a_\alpha = (a^\dagger_+ a_+ + a^\dagger_- a_- + a^\dagger_z a_z)$

⇒ three 1D harmonic oscillators

with commuting creation operators, $a^\dagger_{\alpha=\pm,z}$

In new ± basis: (spherical)

$n_\pm = a^\dagger_\pm a_\pm$, $n_z = a^\dagger_z a_z$

where n_x, n_y, n_z are still good quantum #'s
→ Since $a_\pm^\dagger, a_z^\dagger$ obey same commutation relations as the SHO, $a_\pm^\dagger, a_z^\dagger$ raise N_{tot} by 1
also since $[J_z, a_\pm^\dagger] = \pm\hbar a_\pm^\dagger$, we know $a_\pm^\dagger$ also raise J_z by 1

If $|N_{tot}=0, l=0=m\rangle$ is the lowest energy state...
- ground state
  $N_{tot}=0=n_\pm=n_z$
- first excited state
  $N_{tot}=1=(n_x=1, n_y=0=n_z)$ ← both cyclically permutable
  $=(n_-=1, n_z=0=n_+)$
  $|n_-=1, n_z=0, n_+=0\rangle = a_-^\dagger|0\rangle, \rightarrow |m=-1\rangle$
  $a_+^\dagger|0\rangle, \rightarrow |m=+1\rangle$
  $a_z^\dagger|0\rangle$

But what about l?
$a_+^{\dagger p}|N_{tot}=0, l=0=m\rangle$ is the highest L_z state with $N_{tot}=p$
creation operator applied p-times
  $a_+^{\dagger p}|N_{tot}=0, l=0=m\rangle \propto |N_{tot}=p, l=p=m\rangle$
  $a_-^{\dagger p}|N_{tot}=0, l=0=m\rangle \propto |N_{tot}=p, l=p, \underbrace{m=-p}\rangle$
  remember that $a_-^\dagger$ lowers m!

⟹ The highest l state at $N_{tot}=p$ is $l=p$
for $l\geq 1$
  degeneracy = 3 → $l=1$
  $a_\pm^\dagger|0\rangle \propto |l=1, m=\pm 1\rangle$
  $a_z^\dagger|0\rangle \propto |l=1, m=0\rangle$
  └ it must be true that $a_z^\dagger$ does not modify m because this is the only spherically symmetric state left

⟹ V does not change l values!

Degeneracy of $E=(N_{tot}+3/2)\hbar$:
  $\frac{1}{2}(N_{tot}+1)(N_{tot}+2)$
What is the l, m composition of these degenerate states?
- $l_{max}=N_{tot} \rightarrow l \leq N_{tot}$

• $N_{tot} - \ell$ is even

↓ implies ↓

same parity required

$$\#\text{states} = \sum_{\ell \leq N_{tot}} (2\ell + 1)$$

$$= \sum_{\underbrace{\ell = N_{tot} - 2p}_{[N_{tot}/2] > p \geq 0}} (2N_{tot} + 1 - 4p)$$

$$= \frac{1}{2}(N_{tot}+1)(N_{tot}+2),$$ Just the degeneracy!