10/28/15

Lecture 25 - Principle of Least Action

1. Midterm, Problem 3
2. Path Integral
3. Getting $W(\Delta t)$

1. Midterm: P#3

$$\frac{-\psi''}{2} + \int_{-\infty}^{\infty} dx' \underbrace{\overset{\lambda f(x) f^*(x')}{U(x,x')}}_{\text{usually } V(x)\delta(x-x')} \psi(x') = E\psi(x)$$

a. If λ is < 0, then this has bound states
- change to momentum space (fourier transform); eigenstates should appear

b. Solve this again in real space - follow posted notes
- move $U \rightarrow$ RHS
  $E \rightarrow$ LHS

Hint: $\psi(x) = \underbrace{ae^{i\sqrt{E'}x} + be^{-i\sqrt{E'}x}}_{\varphi \text{ (like the forced harmonic oscillator)}} + \ldots$

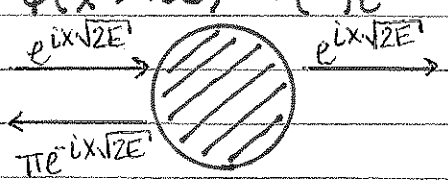
$-\varphi''/2 - E\varphi = 0$ produces a solution

$$\psi(x) = ae^{i\sqrt{E'}x} + be^{-i\sqrt{E'}x} + \beta\left[\int K(x,x';E) f(x')\right] \underbrace{\int dx'' f^*(x'')\psi(x'')}_{= \text{constant}, \ \psi_0; \text{ can be calculated from equation}}$$

c. Follows from interpreting (b) carefully

$\psi(x \rightarrow -\infty) \rightarrow (\ )e^{ix\sqrt{E'}} + (\ )e^{-ix\sqrt{E'}}$

$\psi(x \rightarrow +\infty) \rightarrow (\ )e^{ix\sqrt{E'}} + (\ )e^{-ix\sqrt{E'}}$



pi phase shift given

2. Path Integral

The propagator as a path integral expresses the action over all paths (Feynman's interpretation)

$\longrightarrow$

$$\langle x_N, t_N | x_0, t_0 \rangle = \int \frac{1}{W(\Delta t)^{N-1}}\, dx_1 \dots dx_{N-1}\, \overbrace{\exp\left[\frac{i}{\hbar} \underbrace{\int_{t_0}^{t_N} \mathcal{L}(x,\dot{x})\, d\tau}_{\text{action, } S}\right]}^{\text{phase}}$$

integrating the Lagrangian over all paths

Cross-check Dirac's guess:

$\hbar \to 0 \longrightarrow$ Classical mechanics

$\Rightarrow$ Corresponds to Hamilton's Principle of Least Action

$S[x(\tau)] = \int_{t_0}^{t_N} \mathcal{L}(x, \dot{x})\, d\tau$

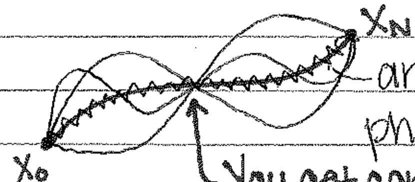
↳ Euler-Lagrange equations ↴

① fix endpoints (x_0, t_0), (x_N, t_N)

② extremize the action

$\delta S|_{\text{endpoints}} = 0$ for $x(\tau) = \underbrace{x_c(\tau)}_{\text{action extremized for the classical path}}$

Feynman states that for quantum mechanics, this is a sum of all paths (i.e., the particle takes <u>all</u> paths)



- around the classical path, the wavefunction's phases stop fluctuating wildly

↳ You get constructive interference around the classical path

Classical Path $\delta S = 0$

$$\int_{x \sim x_c} e^{iS[x(\tau)]/\hbar} \sim e^{i\left(\underbrace{S[x_c]/\hbar}_{\text{classical path}} + \underbrace{\overset{0 \text{ in the classical case}}{\delta S}/\hbar}_{\text{variation in path from classical}}\right)}$$

↳ Classical path dominates the path integral as $\hbar \to 0$
a.k.a. the stationary phase method

- away from the classical path, the phases cancel out because of the wild variation and the probability amplitudes for those paths are very small

*These constructive paths are <u>NOT</u> unique in the quantum mechanical case

3. Getting $W(\Delta t)$

one link of the propagator

$\frac{1}{2}m\dot{x}^2 - V(x)$ for the free particle

$$\langle x_N, t_N | x_{N-1}, t_{N-1} \rangle = \underbrace{\frac{1}{W(\Delta t)}}_{\text{should be independent of } V} \exp\left[\frac{i}{\hbar}\int \mathcal{L}(x,\dot{x})\,d\tau\right]$$

$\rightarrow$ set $V \rightarrow 0$

$$= \frac{1}{W(\Delta t)} \exp\left[\frac{i}{\hbar}\int_{t_{n-1}}^{t_n} \frac{m\dot{x}^2}{2}\,d\tau\right]$$

$|t_n - t_{n-1}|$ assumed a small interval

$$x(\tau) \sim x(t_{n-1}) + \frac{x(t_n) - x(t_{n-1})}{(t_n - t_{n-1})}(\tau - t_{n-1})$$

no assumptions made concerning Δx

$$= \ldots = \frac{1}{W(\Delta t)} \exp\left[\frac{\overset{\sim -1}{im}\overbrace{(x_n - x_{n-1})^2}}{2\hbar\Delta t}\right]$$

approximately Gaussian under $i \sim -1$ assumption

$t_n \rightarrow t_{n-1}$

$\langle x_N, t_N | x_{n-1}, t_{n-1} \rangle \longrightarrow \delta(x_N - x_{n-1})$

Need a value for $W(\Delta t)$ such that

$$\int dx_n \langle x_n, t_{n-1} + \delta | x_{n-1}, t_{n-1} \rangle = 1$$

$$\Longrightarrow W(\Delta t)^{-1} = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}$$

We can now write the full Feynman Path Integral:

product over time intervals

$$\langle x_N, t_N | x_0, t_0 \rangle = \int \underbrace{\prod_j \left\{ dx_j \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{1/2} \right\}}_{\mathcal{D}x} e^{iS[x]/\hbar}$$

representation of the propagator

* this is consistent with / satisfies the Schrödinger Wave equation (2.16 SAK)

$(t_N - t_{N-1})$ assumed infinitesimal

$$\langle x_N, t_N | x_1, t_1 \rangle = \int dx_{N-1} \overbrace{\langle x_N, t_N | x_{N-1}, t_{N-1} \rangle}\langle x_{N-1}, t_{N-1} | x_1, t_1 \rangle$$

$$= \int_{-\infty}^{\infty} dx_{N-1} \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \exp\left[\left(\frac{im}{2\hbar}\right)\frac{(x_N - x_{N-1})^2}{\Delta t} - \frac{iV\Delta t}{\hbar}\right] \langle x_{N-1}, t_{N-1} | x_1, t_1 \rangle$$

define $\xi \equiv x_N - x_{N-1}$

let $x_N \to x$, $t_N \to t + \Delta t$

$$\langle x, t+\Delta t | x_1, t_1 \rangle = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_{-\infty}^{\infty} d\xi \, \exp\left(\frac{i m \xi^2}{2\hbar \Delta t} - \frac{i V \Delta t}{\hbar}\right) \langle x - \xi, t | x_1, t_1 \rangle$$

as $\Delta t \to 0$, $\xi \simeq 0$ provides the major contribution to (i.e., satisfies) the integral

$\to$ expand in ξ and Δt (both small)

$$\langle x, t | x_1, t_1 \rangle + \Delta t \frac{\partial}{\partial t} \langle x, t | x_1, t_1 \rangle$$

$$= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_{-\infty}^{\infty} d\xi \, \exp\left(\frac{i m \xi^2}{2\hbar \Delta t}\right) \left(1 - \underbrace{\frac{V \Delta t i}{\hbar}}_{\text{collect first-order terms in } \Delta t} + \dots\right) \left[\langle x, t | x_1, t_1 \rangle + \left(\frac{\xi^2}{2}\right) \frac{\partial^2}{\partial x^2} \langle x, t | x_1, t_1 \rangle + \dots\right]$$

$$\Delta t \frac{\partial}{\partial t} \langle x, t | x_1, t_1 \rangle = \left(\sqrt{\frac{m}{2\pi i \hbar \Delta t}}\right) \underbrace{\left(\cancel{\sqrt{2\pi}}\right) \left(\frac{i \hbar \Delta t}{m}\right)^{3/2}}_{\int_{-\infty}^{\infty} d\xi \, \xi^2 \exp\left(\frac{i m \xi^2}{2\hbar \Delta t}\right) = \sqrt{2\pi} \left(\frac{i\hbar\Delta t}{m}\right)^{3/2}} \frac{1}{2} \frac{\partial^2}{\partial x^2} \langle x, t | x_1, t_1 \rangle - \left(\frac{i}{\hbar}\right) \Delta t V \langle x, t | x_1, t_1 \rangle$$

By simplifying, it becomes apparent that Feynman's Path Integral $\langle x, t | x_1, t_1 \rangle$ satisfies Schrödinger's time-dependent wave equation:

$$i\hbar \frac{\partial}{\partial t} \langle x, t | x_1, t_1 \rangle = -\left(\frac{\hbar^2}{2m}\right) \frac{\partial^2}{\partial x^2} \langle x, t | x_1, t_1 \rangle + V \langle x, t | x_1, t_1 \rangle$$