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Lecture 24 - Path Integrals

1. Path Integral
2. Propagator as a Path Integral

1. Path Integral

(Feynman's way of viewing the propagator)

Propagator for the TDSE:

$$K(x'', t ; x', t_0) = \langle x'' | \underbrace{e^{-iH(t-t_0)/\hbar}}_{U(t)} | x' \rangle$$

- if the Hamiltonian depends on time:

$$= \langle x'' | T_\tau \exp(-i \int_{t_0}^{t} \mathcal{H}(\tau')d\tau') | x' \rangle$$

└ time-ordered exponential

Propagator as a transition amplitude:

base ket 'x' in the schrodinger picture

$$\begin{cases} |x', t_0\rangle_B = e^{iHt_0/\hbar} |x'\rangle_S \\ |x'', t\rangle_B = e^{iHt/\hbar} |x''\rangle_S \end{cases}$$ Relationships between base-kets and schrödinger-kets

↑ Heisenberg representation treats space & time on the same footing

$K \sim \langle x'', t | x', t_0 \rangle$

↳ the eigenstate of the position operator at time t

$$\hat{x}(t) | x'', t\rangle = \underbrace{x'' | x'', t\rangle}$$

if you measure your position @ t, your wavefunction collapses into the base ket at that time

(x'', t)

(x', t_0)

$\langle x'', t | x', t_0 \rangle$ — Gives the probability amplitude that you will start in state $|x', t_0\rangle$ and be in $|x'', t\rangle$ after some time, t

*this sets up relativity naturally (variations between base kets and observables as reference frames

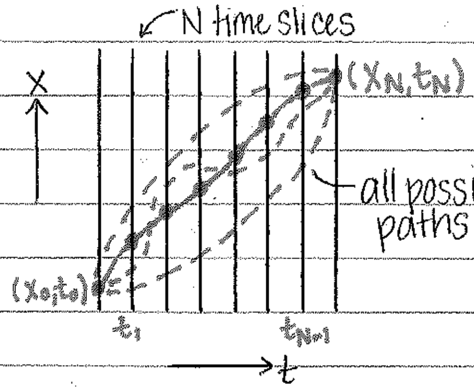
At any given time, the set $\{|x'', t\rangle\}$ forms a complete orthonormal basis → equivalent to saying ↴

$$\int dx'' |x'', t\rangle\langle x'', t| = \mathbb{1}$$

$$\int dx'' |x'', t\rangle (\langle x'', t | \Psi\rangle) = |\Psi\rangle$$

2. Propagator as a Path Integral

t_0 = first time slice @ X_0 $\longrightarrow$ t_N = last time slice @ X_N



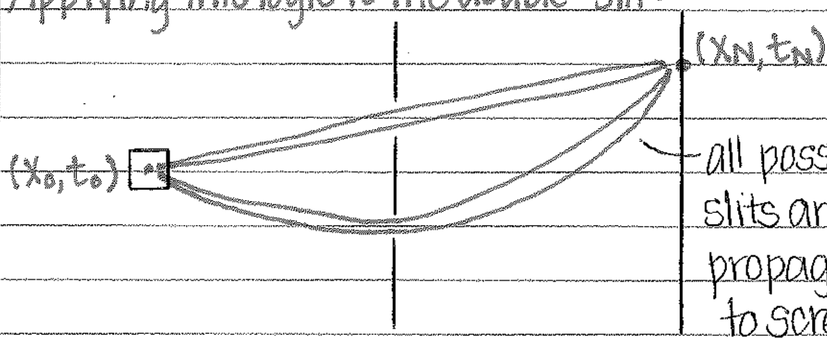
*Need amplitude (probability) of going from $t_0 \rightarrow t_N$

The propagator can be viewed as an integral over all possible paths.

$$\langle X_N, t_N | X_0, t_0 \rangle = \int dx_1 \underbrace{\langle X_N t_N | X_1 t_1 \rangle \langle X_1, t_1 | X_0, t_0 \rangle}_{\text{probability amplitude of going from } |0\rangle \rightarrow |1\rangle}$$

$$= \underbrace{\int dx_1 dx_2 \ldots dx_{N-1}}_{(N-1)\text{-fold integral}} \langle X_N t_N | X_{N-1}, t_{N-1} \rangle \ldots \langle X_2 t_2 | X_1 t_1 \rangle \langle X_1, t_1 | X_0, t_0 \rangle$$

Applying this logic to the double-slit:



all possible paths through both slits are represented by the propagator from source to screen.

$\rightarrow$ Interference, under the logic of the propagator, becomes a natural phenomenon!

But how do we compute this path integral?

Path Integral $\rightarrow$ Action

Dirac: this amplitude is somehow related to the action

$\langle x_j, t_j | x_i, t_i \rangle \sim \exp\left(\frac{i}{\hbar} S[x_i t_i; x_j t_j]\right)$

BUT! $\underbrace{S[x_i t_i; x_j t_j]}_{\text{the classical action}} = \int_{x_i t_i}^{x_j t_j} \mathcal{L}(x, \dot{x})\, dt'$ can only be evaluated over a classical path

-- Feynman proposed a way to express the action over all paths

Feynman's interpretation of Dirac:

$$\langle X_N, t_N | X_{N-1}, t_{N-1} \rangle = \left(\frac{1}{W(\Delta t)}\right) e^{\frac{i}{\hbar} \int_{t_{N-1}}^{t_N} \mathcal{L}(x, \dot{x})\, dt'}$$

where $W(\Delta t)$ is a normalization; independent of the potential $V(x,t)$ and assuming m is constant.

→ Apply the above propagator integral

$$\langle x_N, t_N | x_0, t_0 \rangle = \int dx_1\, dx_2 \ldots dx_{N-1} \langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \ldots \langle x_1, t_1 | x_0, t_0 \rangle$$

$$= \int \frac{1}{W(\Delta t)^{N-1}} dx_1 \ldots dx_{N-1} \exp\left[\frac{i}{\hbar}\left\{\int_{t_0}^{t_1} \mathcal{L}(x,\dot{x})dt + \int_{t_1}^{t_2} \mathcal{L}(x,\dot{x})dt + \ldots + \int_{t_{N-1}}^{t_N} \mathcal{L}(x,\dot{x})dt\right\}\right]$$

$t_N - t_{N-1} = \Delta t$

the action is a superposition along all paths

$$= \frac{1}{W(\Delta t)^{N-1}} \int dx_1 \ldots dx_{N-1} \exp\left(\frac{i}{\hbar} \underbrace{\int_{t_0}^{t_N} \mathcal{L}(x,\dot{x})\, dt}_{S[x(t);\, x_0 t_0;\, x_N t_N]}\right)$$

a functional of the path and the times along it

In the classical limit $\hbar \to 0$

→ saddlepoint/stationary phase method

$$\text{Integral} \sim \left.\frac{\delta S}{\delta x}\right|_{x = \tilde{x}(t)} = 0$$

evaluated along the classical path

⟹ Hamilton's Principle of Least Action