10/19/15

Lecture 21 - Classical Limits & Approximations

1. Classical Limit
2. Homework 6
3. WKB Approximation

1. Classical Limit of the S.W.E.

Need to show that the Schrödinger Wave Equation is consistent with classical mechanics as $\hbar \to 0$

$$\frac{-\hbar^2}{2m}\left[\nabla^2\sqrt{\rho} + \frac{2i}{\hbar}(\nabla\sqrt{\rho})\cdot(\nabla S) - \frac{1}{\hbar^2}\sqrt{\rho}(\nabla S)^2 + \frac{i}{\hbar}\sqrt{\rho}\nabla^2 S\right] + \sqrt{\rho}V$$

$$= i\hbar\left[\frac{\partial\sqrt{\rho}}{\partial t} + \frac{i}{\hbar}\sqrt{\rho}\,\partial_t S\right]$$ note: ρ & S are real (for now)

$\psi = (\sqrt{\rho})e^{iS/\hbar}$

ρ = probability density

Imaginary parts cancel if you take the continuity equation, $\partial_t \rho = -\nabla\cdot(\rho/m\, \nabla S)$, as noticed by David Bohm & deBroglie

S = Hamilton's principle function (classical action $S[x] \sim \int \mathcal{L}\, dt$)

$$\partial_t S = -\Big[\underbrace{V + \frac{1}{2m}(\nabla S)^2}_{\text{usual Hamilton-Jacobi}} \underbrace{- \frac{\hbar^2}{2m}\frac{\nabla^2\rho}{\rho}}_{\text{Bohm's "Quantum Potential", } Q}\Big]$$

as $\hbar \to 0$, $Q \to 0$

Classical analog of time-independent H-J:

if $V(x,t) = V(x)$ (time-independent potential)

$S(x,t) = \underbrace{W(x)}_{\text{Hamilton's characteristic equation}} - Et$

$$\Rightarrow E = \left[V + \frac{1}{2m}(\nabla W)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2\rho}{\rho}\right] \quad (1)$$

$\psi = (\sqrt{\rho})e^{iW/\hbar}\underbrace{e^{-iEt/\hbar}}_{\text{looks like the time-evolution of an eigenstate}}$

Bohmian Mechanics

(taking the Hamilton-Jacobi interpretation seriously)

particle probability = $\rho(x,t)$

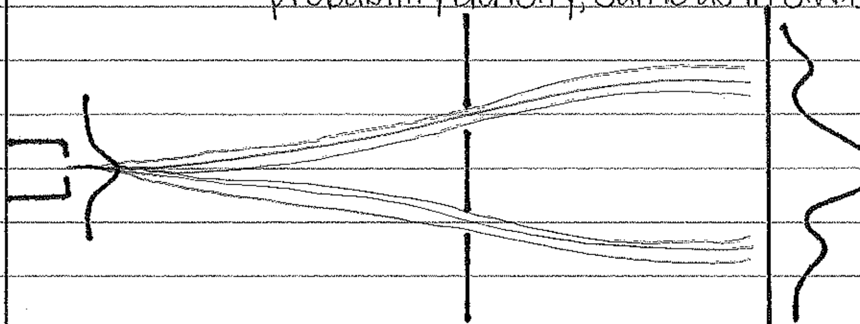
explicit density time-dependence

particle velocity

$$\text{"}v\text{"} = \frac{\vec{J}}{\rho} = \frac{1}{m}\underbrace{\nabla S}_{\text{momentum}}$$

This "velocity" plus the usual Hamilton-Jacobi consistently describes $\rho(x,t)$ and $\vec{J}(x,t)$ ~ current density

└ probability density; same as in S.W.E.



guaranteed to reproduce Q.M. results because it is constructed to do so

Very non-local → every time you make a measurement, ρ instantaneously changes, which changes your quantum potential, which prevents your wavefunction from collapsing.

2. Homework #6

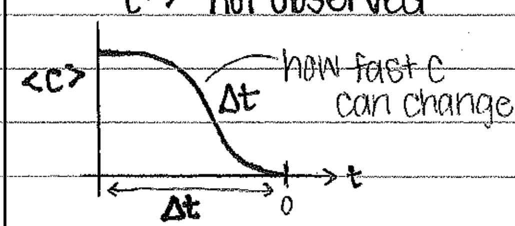
(P1) Energy-Time Uncertainty (2.1 SAK)

If Ψ is an eigenstate, no operator relationship can change.

Really proving: As you vary state Ψ, you can't get C to go from $1 \to 0$ faster than the velocity bound (time-uncertainty principle)

C = "clock" observable → how long it takes a process to happen

t → "not observed"



└ interpreted as a lifetime, etc.

*energy-time is special because time is not really an observable

$$\hat{C} = |\Psi_i\rangle\langle\Psi_i|$$

$$\langle C\rangle = \langle\Psi|\hat{C}|\Psi\rangle$$

$$\Delta E^2 = \langle\Psi|\mathcal{H}^2|\Psi\rangle$$

$$\Delta E \Delta t \gtrsim \hbar$$

└ "velocity bound"

(P2) Coherent States

$|0\rangle \xrightarrow{U_x(\alpha)U_p(\beta)} |\Psi\rangle$ — from HW1(b)

$a|\lambda\rangle = \lambda|\lambda\rangle$ λ related to 0 through a unitary transformation

– apply Campbell-Baker-Hausdorff

$a|0\rangle = 0$

→ You can then calculate the wavefunction of the coherent state

(P3) Schrödinger Wave Equation

"The black hole part of the class"

Proving the "Virial theorem"

$[\hat{x}p_x, \hat{\mathcal{H}}]$, $\hat{\mathcal{H}} = \hat{p}^2/2m + V(x)$

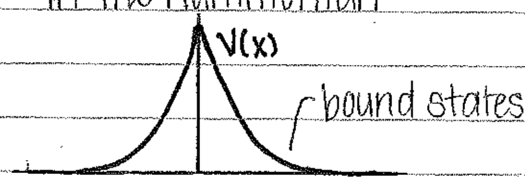
– Show that the KE and the expectation value are equal

$\langle p^2/2m \rangle = \langle x\,\partial_x V(x) \rangle$

(P4) Solve the δ-function Potential

$V(x) = V_0\,\delta(x)$

└ combine this and how you deal with instantaneous changes in the Hamiltonian



$V_{inst.} \rightarrow V_0/2$

Calculate the probability that the particle escapes

3. **WKB Approximation**

Wentzel-Kramers-Brillouin [& Jeffreys] Approximation

In the limit $\hbar \to 0$ in eq. ①, dropping the quantum potential term

(choose 1-dimension) ← this is what makes it an approximation

then $\underbrace{(\partial_x W)}_{\text{was } \Delta W} = \sqrt{2m(E - V(x))}$ ② — gives "face-part" of your wavefunction

↖ E assumed known

→ But what about ρ?

We can get ρ from current conservation

$$\partial_t \rho = -\partial_x\left(\frac{\rho\,\partial_x S}{m}\right) = 0$$ — ρ is time-independent because we chose a time-independent potential

$S = W - Et;\ t=0$

$$= -\partial_x\left(\frac{\rho\,\partial_x W}{m}\right) = 0$$

$$\rho(x) = \frac{\rho_0 m}{\partial_x W} \quad ③$$

Combining equations ② & ③ and reinserting time-dependence →

$$\Psi_{WKB}(x,t) = \frac{\sqrt{\rho_0 m}}{(2m(E-V(x)))^{1/4}} \exp\left[\frac{i}{\hbar}\int^{x} dx' \underbrace{\sqrt{2m(E-V(x'))}}_{p(x')}\right] e^{-iEt/\hbar}$$

$\sim e^{ikx}$

changing the lower limit here just changes the phase

This will allow us to get boundary conditions on Ψ

→ We've assumed E is known, but what WKB is most used for is to choose an appropriate E that satisfies the constraints on the plane wave Ψ

<u>But</u> we can't really send $\hbar \to 0$

define $\underbrace{k(x')}_{\text{inverse length}} \equiv \frac{p(x')}{\hbar} = \frac{\sqrt{2m(E-V(x'))}}{\hbar}$

$$\partial_x^2 V \ll k(x')\,\partial_x V \ ; \quad V = \frac{\hbar^2}{2m}\frac{\partial_x^2\sqrt{\rho}}{\sqrt{\rho}}$$

Actual assumption being made:

This inverse length is assumed to be much much larger than the scale on which the potential is varying.