Brittany's Notes !!
10/14/15

Lecture 19 - Heisenburg's Equation of Motion

1. Heisenburg EOM
2. Harmonic Oscillators

1. The Heisenburg Equation of Motion

Continued from last time...

$$\frac{d}{dt}A^{(H)} = \frac{-i}{\hbar}[A^{(H)}, \underbrace{H^{(H)}}]$$

Hamiltonian in the Heisenburg representation:
$$H^{(H)} = U^{\dagger}(t)H(t)U(t)$$

If the time-independent Hamiltonian or $[H(t), H(t')] = 0$: ← time translation invariant

$$U(t) = e^{-i/\hbar \int_0^t H(t')dt} \text{ and } H^{(H)}(t) = H(t)$$

In general, the Heisenburg Hamiltonian (H-hat) takes a standard functional form: $B_z(t)S_z$ — spin-operator; mag field component

$$\rightarrow H^{(H)}(t) = B_z(t)S_z^{(H)}(t)$$
$$= B_z(t)U^{\dagger}(t)S_zU(t)$$

*This representation is closer to the classical form (if you take $[A,B] \rightarrow i\hbar\{A,B\}$ it's exactly the same)

$$\frac{d}{dt}A(p,q) = \frac{\partial A}{\partial p}\underbrace{\frac{dp}{dt}} + \frac{\partial A}{\partial q}\underbrace{\frac{dq}{dt}}$$ — assumed no explicit time-dependence in A

Use Hamilton's equations

$$= -\frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial q}\frac{\partial H}{\partial p}$$
$$= -\{H,A\} = \{A,H\}$$ ✓ Classical form
$$= -i/\hbar[A,H]$$, Heisenburg EOM (recovered)

Example: The free particle

$$H = \vec{p}^2/2m$$

Note: everything here in Heisenburg representation -- H superscript neglected for ease of writing

- $$\frac{d\vec{p}}{dt} = \frac{-i}{\hbar}\left[\vec{p}, \frac{\vec{p}^2}{2m}\right] = 0$$

Whenever an observable $A^{(H)}$ commutes with the Hamiltonian, we know $A^{(H)}$ is a constant of motion

→ conservation of momentum

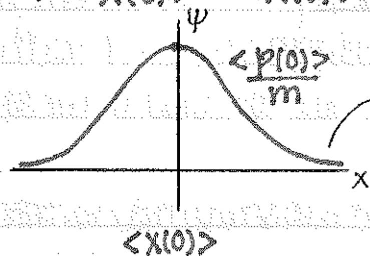
$\cdot \dfrac{d\vec{x}}{dt} = \dfrac{-i}{\hbar}\left[\vec{x}, \dfrac{\vec{p}^2}{2m}\right]$ *Recall from HW#4: computed the commutator of $[x, G(p)] = i\hbar\,\partial G/\partial p$

$= \cancel{\dfrac{1}{i\hbar}}\,\dfrac{1}{\cancel{2}m}\,(\cancel{i\hbar}\,\cancel{2}\vec{p}) = \dfrac{\vec{p}}{m}$ (velocity, as expected)

$= \dfrac{\vec{p}(0)}{m}$ – since p is constant

$\Rightarrow \vec{x}(t) = \vec{x}(0) + \vec{p}(0)t/m$ – like classical uniform linear motion

$\hookrightarrow \langle \vec{x}(t) \rangle = \langle \vec{x}(0) \rangle + \frac{t}{m}\langle \vec{p}(0) \rangle$



The "spread" in the expectation value of the wavefunction

General uncertainty:

$\underbrace{\langle (\Delta x(t))^2 \rangle}_{\text{defined as variance}} \langle (\Delta x(0))^2 \rangle \geq \frac{1}{4}\left|\langle \underbrace{[x(t), x(0)]}_{\frac{t}{m}i\hbar} \rangle\right|^2$

$= \langle x(t)^2 \rangle - \langle x(t) \rangle^2$

$= \dfrac{t^2\hbar^2}{4m^2}$

$\Rightarrow \langle \Delta x(t)^2 \rangle \geq \dfrac{t^2\hbar^2}{4m^2 \langle \Delta x(0)^2 \rangle}$ This relation implies that even if the particle is well localized at $t=0$, its position becomes more and more uncertain with time

Now, add a potential: $\mathcal{H} = p^2/2m + V(\vec{x})$ – time-independent

$\cdot \dfrac{d\vec{x}}{dt} = \dfrac{-i}{\hbar}\left[\vec{x}, \dfrac{\vec{p}^2}{2m} + V(\vec{x})\right] = \dfrac{\vec{p}}{m}$ (unchanged)

$\cdot \dfrac{d\vec{p}}{dt} = \dfrac{-i}{\hbar}\underbrace{\left[\vec{p}, \dfrac{\vec{p}^2}{2m} + V(\vec{x})\right]}_{[\vec{p}, V(\vec{x})] = -i\hbar\vec{\nabla}V(\vec{x})} = \dfrac{1}{\cancel{i\hbar}}(-\cancel{i\hbar}\vec{\nabla}V(\vec{x}))$

$= -\vec{\nabla}V(\vec{x})$

$m\dfrac{d^2\vec{x}}{dt^2} = \dfrac{d\vec{p}}{dt} = -\vec{\nabla}V(\vec{x})$ – QM analog of Newton's 2nd law

└ this cannot be solved as the commutator of $[x(t), x(t')] \neq 0 \left(= \dfrac{-i\hbar t}{m}\right)$

Ehrenfest's Theorem

- Take the expectation value of both sides w.r.t. a Heisenburg state ket that is time-invariant -

$$m\frac{d^2}{dt^2}\langle\vec{x}\rangle = \frac{d\langle\vec{p}\rangle}{dt} = -\langle\vec{\nabla}V(\vec{x})\rangle$$

* unlike the vector form above that so resembles Newton's 2nd law, this theorem in expectation value form is valid for both the Heisenburg <u>and</u> the Schrödinger representations

Note that $\hbar$ has disappeared → the center of the wave packet moves like a classical particle subjected to $V(\vec{x})$

Transition Amplitudes & Base Kets

(time evolution)

In the Heisenburg representation, $A^{(H)}(t)$ evolve in time but states $|\Psi\rangle$ are stationary $|\Psi\rangle \equiv |\Psi(0)\rangle$

└ This is <u>NOT</u> to imply that all kets are stationary here

→ Base Kets DO evolve in the Heisenburg picture

$|x(t)=x_0\rangle$ = base ket, $|A^{(H)}(t)=a\rangle$

*Need to distinguish the behavior of state kets from that of base kets

Heisenburg equation of motion

$\Rightarrow A^{(H)}(t)|A^{(H)}(t)=a\rangle = a|A^{(H)}(t)=a\rangle$

time evolution of base kets

$\hookrightarrow U^{\dagger}(t)AU(t)U^{\dagger}(t)|A=a\rangle = aU^{\dagger}(t)|A=a\rangle$

└ acting on $= a|A=a\rangle$

$= a|A^{(H)}(t)=a\rangle$

$U^{\dagger}(t)|A^{(H)}(0)=a\rangle = |A^{(H)}(t)=a\rangle$

↑ reverse from Schrödinger representation

Transition Amplitudes

$\langle\underbrace{\hat{x}(t)=x'}_{\text{base bra}}|\hat{x}(0)=x''\rangle$

- For a system prepared @ $t=0$ to be in an eigenstate of observable $\hat{x}$ w/ eigenvalue x'', the Transition Amplitude is the probability amplitude that at some later time t, the system will be found in an eigenstate of observable $\hat{x}$ w/ eigenvalue x'

Where $\langle x'|U(t,0)|x''\rangle$ is the transition amplitude for "going" from state $|x''\rangle$ to state $|x'\rangle$

2. Harmonic Oscillators

angular frequency $\omega = \sqrt{k/m}$

Basic Hamiltonian of the S.H.O. : $\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$

$[\hat{p}, \hat{x}] = -i\hbar$

let $\hat{p} \rightarrow \hat{p}\lambda$
$\hat{x} \rightarrow \hat{x}/\lambda$ $\Rightarrow \mathcal{H} = \frac{\hat{p}^2\lambda^2}{2m} + \frac{m\omega^2\hat{x}^2}{2\lambda^2}$

We want the coefficients of each term to be the same

$\frac{\lambda^2}{2m} = \frac{m\omega^2}{2\lambda^2} \longrightarrow \lambda = (m^2\omega^2)^{1/4}$
$= \sqrt{m\omega}$

$\mathcal{H} = \frac{\omega}{2}(\hat{p}^2 + \hat{x}^2)$
$= \frac{\omega}{2}\left[(\hat{x}+i\hat{p})(\hat{x}-i\hat{p}) + i\underbrace{[\hat{p},\hat{x}]}_{-i\hbar}\right]$

define: $\frac{1}{\sqrt{2}}(\hat{x}-i\hat{p}) \equiv a^{\dagger}$, the raising operator such that
$= \hbar\omega(a^{\dagger}a + \frac{1}{2})$

$a \equiv$ the lowering operator

$[a^{\dagger}, a] = \frac{1}{2\hbar}[\hat{x}-i\hat{p}, \hat{x}+i\hat{p}] = \frac{1}{\cancel{2}\hbar}(\cancel{2}i)[\hat{x},\hat{p}]$
$= \frac{1}{\hbar}(i\hbar)$
$= -1$