# Lecture 16 - The Density Matrix

1. Continued - Density Matrix
2. Bloch Spheres

## 1. The Density Matrix

(HW5 Problem 1, Part 1)

A, B = composite system

$$\underbrace{\{|a\rangle\}_{a=1}^{N_A} \otimes \{|b\rangle\}_{b=1}^{N_B}}_{\text{set of composite eigenkets}} = |a\rangle \otimes |b\rangle \leftarrow \text{basis in tensor product space}$$

$|\Psi\rangle = \sum_{a,b} C_{a,b} |a\rangle \otimes |b\rangle \leftarrow$ composite pure state between A&B

↑ any wavefunction in this composite system

Observables in A $= P_A \otimes \mathbb{I}$

$$\langle P_A \rangle = \langle \Psi | P_A \otimes \mathbb{I} | \Psi \rangle$$

$$= \ldots = \sum_{a,a'} \underbrace{\left( \sum_b C^*_{ab} C_{a'b} \right)}_{\rho^{(A)}_{aa'} = \text{Reduced density matrix}} \langle a | P_A | a' \rangle$$

$$= \mathrm{Tr}\{\rho^{(A)} P_A\}$$

mixed state (created on B, but with B now not observable)

Density matrix $\rho^{(A)} = \sum_j P_j |\psi_j^{(A)}\rangle\langle\psi_j^{(A)}|$

In a "pure state", one of these terms would be 1 and the rest would be zero

→ The density matrix combines the quantum states with what you know from statistical mechanics

Note: "pure state" $\longleftrightarrow |\psi\rangle$

$\langle P \rangle = \langle \psi | P | \psi \rangle$

$\Rightarrow \rho = |\psi\rangle\langle\psi| \rightarrow \mathrm{Tr}\{\rho P\} = \langle \psi | P | \psi \rangle$

*Entanglement is analogous to creating classical randomness!*

e.g. You get all of your values for $P_j$ by tossing a coin

## Time Evolution

(HW5 Problem 1, Part 2)

If A&B are far away (i.e. non-interacting), the Hamiltonian takes the form:

$\hat{H} = H_A \otimes \mathbb{I} + \mathbb{I} \otimes H_B$

└ interaction required to produce entanglement

This yields the time evolution operator

$\rightarrow U(t) = e^{-iHt/\hbar} = U_A \otimes U_B$

$= e^{-iH_A t/\hbar} \otimes e^{-iH_B t/\hbar}$

$|\psi(t)\rangle = \sum_{a,b} C_{ab}\, e^{-iE_A t/\hbar} e^{-iE_B t/\hbar} |a\rangle \otimes |b\rangle$

Properties of the Density Matrix

(called a matrix, really an operator)

$\langle P \rangle = Tr\{\rho P\}$

└ for usual "pure states" $|\psi\rangle$

*$|\psi\rangle$ is like classically defining $(\hat{x}, \hat{p})$; $\rho$ is like the phase distribution in stat. mech.

$\rho$ is Hermitian

$\rightarrow$ then $\rho = |\psi\rangle\langle\psi|$

Because $\rho$ is Hermitian, we can diagonalize it

$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

└ a complete orthogonal basis

$\langle P \rangle = \sum_j p_j \langle\psi_j| P |\psi_j\rangle$

$\rightarrow$ apply "coin-toss" interpretation $\rightarrow$

$p_j \geq 0 \longleftrightarrow$ probabilities

$\sum_j p_j = 1 \Longleftrightarrow Tr\{\rho\} = 1$ (positive definite)

- for $\rho_1, \rho_2$, the resulting expectation value would have to be an average

$\rho = \lambda\rho_1 + (1-\lambda)\rho_2$

└ proportion of the whole ensemble represented by $\rho_1$

Compactly: The density matrix is Hermitian, positive definite, and is represented by an operator $\rho$ with $Tr\{\rho\} = 1$

2. Bloch Spheres.

(for a 2-state, finite-dimensional system)

Operators form a Hilbert Space of their own

$\mathcal{H}$ = Hilbert space, $\mathcal{H}'$ = set of operators on $\mathcal{H}$

└ this is a complex vector space $\rightarrow$ can be thought of like bras and kets

$P_1, P_2 \in \mathcal{H}' \rightarrow (c_1 P_1 + c_2 P_2) \in \mathcal{H}'$

(linear combinations live in the same Hilbert space)

$\sim \langle P_1, P_2 \rangle = \mathrm{Tr}\{P_1^\dagger P_2\} \geq 0$

Also linearity, etc., apply in the operator space

If $\mathcal{H}$ = 2-state system in the $|\pm; z\rangle$ basis:

Claim there is a Pauli matrix

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = C_0 \frac{\mathbb{1}}{\sqrt{2}} + C_x \frac{X}{\sqrt{2}} + C_y \frac{Y}{\sqrt{2}} + C_z \frac{Z}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} C_0 + C_z & C_x + iC_y \\ C_x - iC_y & C_0 - C_z \end{pmatrix}$$

the standard Pauli matrices

where $1/\sqrt{2}\{\mathbb{1}, X, Y, Z\}$ form an orthonormal basis

This allows us to write

$$\mathrm{Tr}\left\{\frac{\mathbb{1}^2, X^2, Y^2, Z^2}{2}\right\} \sim \langle P|P \rangle = 1$$ Normalization contition $\mathrm{Tr}^2 = 1$

$$= \mathrm{Tr}\{\tfrac{1}{2}\mathbb{1}\} = 1$$

and also

$$\mathrm{Tr}\left\{\frac{\mathbb{1}}{\sqrt{2}} \frac{X}{\sqrt{2}}\right\} = \frac{1}{2}\mathrm{Tr}\{X\} = 0$$

$\Rightarrow \langle P|P' \rangle = 0$ for $P, P'$ different Pauli matrices

$$C_0 = \mathrm{Tr}\left\{P \frac{\mathbb{1}}{\sqrt{2}}\right\}, \quad C_x = \mathrm{Tr}\left\{P \frac{X}{\sqrt{2}}\right\}, \text{ etc.}$$

$\rightarrow$ Apply this to the density matrix; $\rho$ is Hermitian

$\rho = \rho^\dagger \Longleftrightarrow C_0, C_x, C_y, C_z$ are real here

$\mathrm{Tr}\{\rho\} = 1 \Longrightarrow C_0\sqrt{2} = 1$

$C_0 = 1/\sqrt{2}$

define $a_j \equiv c_j\sqrt{2}$

$$\rho = \frac{\mathbb{1}}{2} + \frac{1}{2}(a_x X + a_y Y + a_z Z)$$

$$\mathrm{DM} \longleftrightarrow (a_x, a_y, a_z) \equiv \underbrace{\text{3-Vector}}_{\text{real!}}$$

General note:

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

$\underbrace{0 \leq p_j \leq 1}_{\text{because they're probabilities}} \implies Tr\{\rho^2\} = \sum_j p_j^2 \leq 1$

└ $Tr\{\rho^2\} = 1$ only for pure states
(non-pure $\sum_j p_j = 1$, so for each $p_j < 1$ it becomes even smaller when squared)

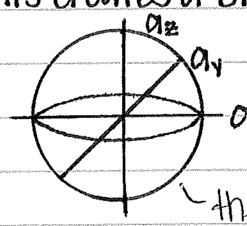
For a 2-state system:

$$Tr\{\rho^2\} = \frac{1}{4}\left[Tr\{\mathbb{1}^2\} + Tr\{X^2\} a_x^2 + Tr\{Y^2\} a_y^2 + Tr\{Z^2\} a_z^2\right]$$

$$= \frac{1}{2}(1 + a_x^2 + a_y^2 + a_z^2) \leq 1$$

$$\implies a_x^2 + a_y^2 + a_z^2 \leq 1$$

This creates a Bloch Sphere!




└ the surface, with values of 1, is composed of pure states

Exercise:

$\rho = |+z\rangle\langle +z| \longleftrightarrow (0,0,1)$
└ North Pole

$= |-z\rangle\langle -z| \longleftrightarrow (0,0,-1)$
└ South Pole

$= |+x\rangle\langle +x| \longleftrightarrow (0,1,0)$, etc.

Totally mixed state $= \dfrac{|+z\rangle\langle +z| + |-z\rangle\langle -z|}{2} = \dfrac{\mathbb{1}}{2}$

$\implies (0,0,0)$, the Origin

*the totally mixed state does not care about direction at all