10 / 05 / 15

Lecture 15 – Measurement & Entanglement

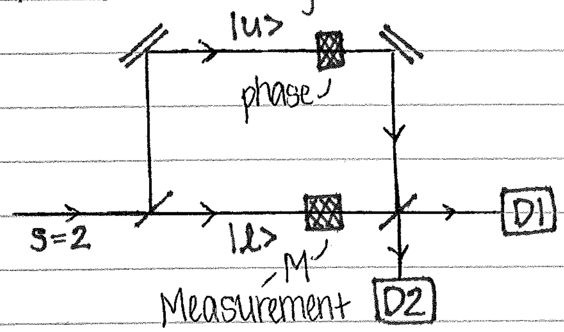
Brittany's Notes

1. Measurement from Entanglement Perspective
2. Density Matrices
3. Reduced Density Matrix

1. Measurement from the Entanglement Perspective

→ Back to interferometer scheme →

Particle with magnetic moment (spin 2)



Interference $P(D1) \propto \cos(\varphi)$

M = measurement
- → off: interference
- → on: classical (no φ dependence)

Quantum description for M:
(in terms of entanglement)

$$M = \text{spin-}\tfrac{1}{2} \left(|+x\rangle = \frac{|+z\rangle + |-z\rangle}{\sqrt{2}} \right) \text{ state}$$

$$= \text{pointer detector} \begin{cases} |+x\rangle \rightarrow |u\rangle \\ |-x\rangle \rightarrow |\ell\rangle \end{cases}$$

Hamiltonian –

constant ↴ ↱ S_z spin operator for M

$$\text{add } \mathcal{H}_{inf} = g\, V \otimes Z$$

↳ $|\ell\rangle\langle\ell|$, spin of particle

$$U(t) = e^{-iH_{inf}t/\hbar}$$

→ choosing $gt/\hbar = \pi$ (maximally entangled state) →

$$= (|u\rangle\langle u|) \otimes \mathbb{1} + (|\ell\rangle\langle\ell|) \otimes (|+z\rangle\langle+z| - |-z\rangle\langle-z|)$$

$$U(t) \underbrace{\left(\frac{|u\rangle e^{i\varphi} + |\ell\rangle}{\sqrt{2}} \right) \otimes |+x\rangle}_{\text{before time evolution}}$$

$$= \left[|u\rangle e^{i\varphi} \otimes |+x\rangle + |\ell\rangle \otimes \underbrace{|-x\rangle}_{1/\sqrt{2}\,(|+z\rangle - |-z\rangle)} \right] \frac{1}{\sqrt{2}} \text{ (by linearity of the tensor product)}$$

$$= \frac{[e^{i\varphi}|u\rangle \otimes |+x\rangle + |\ell\rangle \otimes |-x\rangle]}{\sqrt{2}}$$

if $t \neq {}^{\pi}\!/_{g}$:

$|-x\rangle \to \alpha|-x\rangle + \beta|+x\rangle$, "Partial or Weak Measurement"

* before, $|u\rangle \to |+x\rangle$ and $|\ell\rangle \to |-x\rangle$, but in the case of weak measurement $|+x\rangle$ & $|-x\rangle$ are not completely separated

$$\Rightarrow |\psi\rangle \xrightarrow[\text{splitter}]{\text{beam}} \frac{1}{2}\left[e^{i\varphi}\left(\frac{|u\rangle + |\ell\rangle}{\sqrt{2}}\right) \otimes |+x\rangle + \left(\frac{|u\rangle - |\ell\rangle}{\sqrt{2}}\right) \otimes |-x\rangle\right]$$

$\hookrightarrow$ Measure at D1

$$\frac{1}{2}\left[e^{i\varphi}|u\rangle \otimes |+x\rangle + |u\rangle \otimes |-x\rangle\right] = \frac{1}{2}\left(e^{i\varphi}|+x\rangle + |-x\rangle\right) \otimes |u\rangle$$

Probability $\to$ NORM $= 1/4(1+1) = 1/2$

Without measurement:

$t = 0 \Rightarrow \beta = 1$

$$\to \underset{\text{(interference)}}{1/4\,|e^{i\varphi}|+x\rangle + |+x\rangle|^2} = \left(1 + \frac{\cos(\varphi)}{2}\right)$$

2. Density Matrices

Classical vs. Quantum Uncertainty

Classical randomness $\nearrow$ $|+z\rangle$ or $|-z\rangle$; $\searrow$ $|+x\rangle$ or $|-x\rangle$

To represent this classical randomness, we can use a density matrix

For a set of states $\{|\psi_\alpha\rangle\}$ with classical probability p_α

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

then for some observable, $\hat{A}$

expectation value

$$\langle\hat{A}\rangle = \sum_\alpha p_\alpha \langle\psi_\alpha|\hat{A}|\psi_\alpha\rangle$$

Conveniently: $\mathrm{Tr}(\rho\hat{A}) = \langle\hat{A}\rangle$

$\llcorner \rho = |\psi\rangle\langle\psi|$ such that $\mathrm{Tr}\{\rho\hat{P}\} = \langle\psi|\hat{P}|\psi\rangle$

$|+x\rangle$ – Quantum uncertainty $\to \rho = |+;x\rangle\langle +;x|$

$$= \frac{1}{2}\left[|+z\rangle\langle +z| + |-z\rangle\langle -z| + |+z\rangle\langle -z| + |-z\rangle\langle +z|\right]$$

$\underset{+1/2}{|+z\rangle}\,;\ \underset{+1/2}{|-z\rangle} \longrightarrow \rho = \frac{1}{2}\left[|+z\rangle\langle +z| + |-z\rangle\langle -z|\right] = \frac{1}{2}$

$\ldots = \frac{1}{2}\left[|+x\rangle\langle +x| + |-x\rangle\langle -x|\right]$

$\longrightarrow$

Traditional Measurement

(last problem, HW5)

$$\text{Start in } |\psi\rangle \rightarrow \text{measure } S_z \begin{cases} \nearrow |+z\rangle \text{ with probability } |\langle +z|\psi\rangle|^2 \\ \searrow |-z\rangle \text{ with probability } |\langle -z|\psi\rangle|^2 \end{cases}$$

so $\Rightarrow \rho = \sum_{S_z=\pm} |S_z\rangle\langle S_z| \left(|\langle S_z|\psi\rangle|^2\right)$

* Making a density matrix $\longleftrightarrow$ entanglement

3. Reduced Density Matrix

A, B = Composite system; two Hilbert spaces

$\mathcal{H}^A \otimes \mathcal{H}^B$

$\{|a\rangle\}$, $\{|b\rangle\}$ set of composite eigenkets

$|\psi\rangle = \sum_{a,b} C_{ab} |a\rangle \otimes |b\rangle$

└ most general wavefunction; composite pure state between A & B

for observable $P_A = P_A \otimes \mathbb{1} \Rightarrow \langle\psi|P_A|\psi\rangle$

$$\langle P_A \rangle = \sum_{a,b,a',b'} C^*_{ab} C_{a'b'} \langle a,b| P_A \otimes \mathbb{1} |a'b'\rangle$$

$$= \ldots = \sum_{aa'} \underbrace{\sum_b C^*_{ab} C_{a'b}}_{\rho^{(A)}_{aa'} = \text{Reduced density matrix}} \langle a|P_A|a'\rangle$$

$$= \mathrm{Tr}(\rho^{red} P_A)$$

└ only if $\rho^{red} = \sum_b C^*_{ab} C_{a'b} |a'\rangle\langle a|$